A Cram on the Prolonged Network of Star Grid

K. Thiagarajan, P. Mansoor

Abstract: Nowadays graph theory has indispensable role in many applications of networking. The graph structures are very useful in the modelling of pair wise relations between various nodes of a network. The star graph is a network graph for which various results have been already established. The introduction of semi nodes to each edge in a non trivial star graph $S_n$, on $n \geq 2$ nodes; will result in a new network graph, denoted by $S'_n$. This paper attempts to study some properties of the graph $S'_n$ in terms of graph theoretic parameters, namely maximum degree $\Delta$, chromatic number $\chi$, harmonious chromatic number $h$, domination number $\gamma$.

Index Terms: Expanded Star Graph, Chromatic Number, Harmonious Chromatic Number, Domination Number and Semi Node.

I. INTRODUCTION

In this paper, assume that all the graphs to be considered are connected, loop less and undirected. We begin with basic definitions.

A graph $G$ depicts of a non-empty set $V$ of points, called nodes, and a set $E$ of two point subsets of $V$, called edges connecting pairs of nodes. The number of edges incident with a node is known as the degree of the node. A walk in a graph $G$ is an alternating sequence of distinct nodes and edges which starts and terminates with a node. If no edges are repeating, then the walk is called a trail. In a trail if the starting node and the terminating node is same, then the trail is a circuit. If no vertices are repeating in a walk, then it is called a path. In a path if the starting node and the terminating node are same, then the path is a circuit. For a star graph $S_n$, on $n \geq 2$ nodes, since $\Delta = n - 1$, the chromatic number $\chi(G)$ of the graph $G$ is the minimum number of colours required for colouring $G$ properly.

A harmonious colouring of a graph $G$ is a special type of proper colouring of $G$ such that the colour pairs never repeats. The minimum number of colours required for a harmonious colouring of $G$ is called the harmonious chromatic number of $G$, which is denoted by $h(G)$.

A semi graph $G$ is a pair $(V, E)$ where $V$ is a non-empty set whose elements are called nodes of $G$ and $E$ is the set of edges whose elements are the tuples of distinct nodes for various $n \geq 2$, satisfying the following two conditions:

(i) Any two edges have at most one node in common.
(ii) Two edges $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_l)$ are considered to be equal if and only if $k = l$ and either $u = v$ or $u = v_{i+1}$ for $i = 1, 2, 3, \ldots, n$.

Thus, the edges $(u_1, u_2, \ldots, u_k)$ and $(u_1, u_2, \ldots, u_l)$ are the same.

When splicing a graph $G$, the new nodes obtained are called semi nodes and the new edges formed by the decomposition of edges are called semi edges.

II. HARMONIOUS CHROMATIC NUMBER OF A STAR GRAPH

$S_n, n \geq 2$

Definition: A graph $G$ having $n - 1$ pendant nodes and exactly one node of degree $n - 1$ is said to be a star graph. It is denoted by $S'_n$, where $n > 0$. Here we consider only the non trivial star graphs $S'_n, n \geq 2$.

For a star graph $S_n$, we have the following:

- The number of nodes $|V| = n$.
- The number of edges $|E| = n - 1$.
- Maximum vertex degree $\Delta = n - 1$.
- Minimum vertex degree $\delta = 1$.
- The domination number $\gamma = 1$.
- The chromatic number $\chi = 2$.

By the definition of a star graph $S_n$, since $n - 1$ vertices are adjacent to a single vertex, it requires $n - 1 + 1 = n$ colours for the harmonious colouring. It shows that the harmonious chromatic number $h = n$ for a star graph $S'_n, n \geq 2$.

Example: Consider the star graph $S'_2$. 

Revised Manuscript Received on December 22, 2018.

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Here \(|V| = 4\), \(|E| = 4 - 1 = 3\), \(|\Delta| = 4 - 1 = 3\), \(\delta = 1\), \(\gamma = 1\), \(\chi = 2\) and \(h = 4\).

III. EXPANDED STAR GRAPH

Definition: An expanded star graph denoted by \(S^t_n\) is a graph obtained from a star graph \(S_n\) on \(n \geq 2\) vertices by introducing semi nodes to each of the edges.

The number of nodes \(|V| = n + (n - 1) = 2n - 1\), since we introduce a node for each of the \(n - 1\) edges and \(|E| = n - 1 + 2(n - 1) = 3n - 3\), as \(n - 1\) edges gives two more new edges.

In an expanded star graph \(S^t_n\), all the vertices are adjacent to a single node. Therefore the domination number \(\gamma(S^t_n)\) = 1.

Each edge of \(S^t_n\) will form a cycle of length three in \(S^t_n\) and hence the chromatic number \(\chi(S^t_n) = 3\). Since \(\chi(S^t_n) = 2\) for the star graph \(S_n\), we have \(\chi(S^t_n) = 3 = 2 + 1 = \chi(S_n) + 1\). That is, \(\chi(S^t_n) = \chi(S_n) + 1\).

Proposition 1: For any expanded star graph \(S^t_n\) on \(n \geq 2\) nodes, the maximum degree is \(\Delta(S^t_n) = 2(n - 1)\) and the minimum degree is \(\delta(S^t_n) = 2\).

Proof: Let \(S^t_n\) be a star graph on \(n \geq 2\) nodes. Suppose that \(u\) is the maximum degree node with \(\Delta(S^t_n) = \deg(u) = n - 1\) and \(e = (u, v)\) is the edge connecting \(u\) and \(v\), a pendant node of \(S^t_n\). Let \(w\) be a semi node introduced on the edge. Then \(e\) splits into two semi edges \(e_1 = (u, w)\) and \(e_2 = (w, v)\) and this will contribute an additional count of one into the degree of the node \(u\). Similarly, for each of the remaining edges we obtain additional count of one in to the degree of \(u\).

Thus, \(\deg(u) = n - 1 + (1 + 1 + ... + 1) = n - 1 + (n - 1)\).

That is, \(\deg(u)\) increases by \(n - 1\) on the introduction of semi nodes to each edge of \(S^t_n\). But each of the pendant nodes of \(S^t_n\) will become a node of degree two of \(S^t_n\). Therefore we can conclude that \(u\) is the maximum degree node of the expanded star graph \(S^t_n\), \(n \geq 2\).

Hence the maximum degree is \(\Delta(S^t_n) = \deg(u) = 2(n - 1)\) and the minimum degree is \(\delta(S^t_n) = 2\).

Proposition 2: For any expanded star graph \(S^t_n\), \(n \geq 2\), the harmonic chromatic number \(h(S^t_n) = 2n - 1\).

Proof: For any graph \(G\), we have \(\Delta(G) + 1 \leq \Delta(G) \leq |V(G)|\) which implies that \(\Delta(S^t_n) + 1 \leq \Delta(S^t_n) \leq |V(S^t_n)|\).

i.e., \(2n - 2 + 1 \leq h(S^t_n) \leq 2n - 1\)

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Remarks: From the above result, we have

(i) \(h(S^t_n) = 2h(S_n) - 1\), since \(h(S^t_n) = n \Rightarrow 2h(S^t_n) - 1 = 2n - 1\).

(ii) \(h(S^t_n) = \Delta(S^t_n) + m(S^t_n)\), since \(\Delta(S^t_n) = h(S^t_n)\) and \(m(S^t_n) = |E(S^t_n)| = n - 1\).

(iii) \(h(S^t_n) = \Delta(S^t_n) + \gamma(S^t_n)\), since \(\Delta(S^t_n) = 2(n - 1) = 2n - 2\) and \(\gamma(S^t_n) = 2\).

Example: Consider the expanded star graph \(S^t_4\).

Here \(|V| = 2 \times 4 - 1 = 7\), \(|E| = 3 \times 4 - 3 = 9\), \(\gamma = 1\), \(\chi = 3\), \(\Delta = 3(4 - 1) = 6\), \(\delta = 2\) and \(h = 2 \times 4 - 1 = 7\).

Theorem: Every expanded star graph \(S^t_n\), \(n \geq 2\) is Eulerian.

Proof: Let \(S^t_n\), \(n \geq 2\) be an expanded star graph with \(u\) as the maximum degree vertex. Then \(\deg(u) = 2(n - 1)\), which is even. All the other vertices are of degree 2. That is, all the vertices of \(S^t_n\), \(n \geq 2\) are of even degree and hence it is Eulerian.

Theorem: The expanded star graph \(S^t_n\), \(n \geq 2\) is not Hamiltonian.

Proof: In the expanded star graph \(S^t_n\), \(n \geq 2\), there exists a common node which is adjacent to all the other nodes. All the cycles formed in \(S^t_n\), \(n \geq 2\) will intersect at this node and therefore we cannot find a circuit without the repetition of this common node. Hence it is not Hamiltonian.

IV. CONCLUSION

In this paper we introduced the expanded star graph \(S^t_n\), which is a network graph obtained from the star graph by introducing semi-nodes in each edges and discussed some properties of expanded star graph.

ACKNOWLEDGMENT

It’s our great pleasure to thank Dr. Ponnammal Natarajan, Former Director-Research, Anna University, Chennai, India, for her valuable ideas and discussions for preparing this research paper.

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