A Study On Different Approaches for Solving an Infinite Series

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Abstract: ***In various science and engineering experiments, even in day to day life, we come across various integers or some rational numbers which leads to use the real and complex numbers for the theoretical study to construct a mathematical model. Infinite series are essential for obtaining the solutions either exactly and approximately in solving differential equations that arise in every branch of science and engineering. Also many functions that exist in engineering are defined only through infinite series, and it is important to find the convergence of these series both theoretical and approximate solution of the function.***

Infinite series are used to approximate functions and to calculate transforms in science and engineering. In this paper, the comparison of the convergence of the series with various methods was explained with some specific examples and solution of the infinite series $\sum_{n=1}^{\infty}$ ***by different methods have been discussed.***

Index Terms: ***Infinite series, convergence, partial.***

I. INTRODUCTION

Infinite series plays an important role in determining the solutions of mathematical, scientific and engineering problems [1]. The speed of convergence of sequences is necessary in predicting theory of transformations. The study on faster convergence especially the convergence of partial sums of series to find exact values for a large class of convergent series were made in [2], [3].

The term finite series is defined by the terms of a finite sequence. Consider $[k^2+1]_{k=1}^{8} = \{2,5,10,17,26,37,50,65\}$, then the finite series is obtained by adding all the terms in the sequence together $(i.e)\{2+5+10+17+26+37+50+65\}$.Here the sum of the series is 212. In the same manner, an infinite series is obtained by summing the terms of an infinite sequence. For example, consider $\left[\frac{1}{3^k}\right]_{k=1}^{\infty} = \left\{\frac{1}{3},\frac{1}{9},\frac{1}{27},\frac{1}{81},\dots\right\}$, then the sum of the infinite series is calculated by $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$, and it goes on, but the sum cannot be determined. For better understanding, consider the infinite series $\sum_{k=0}^{\infty}\frac{1}{k!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots = 1+1+\frac{1}{2}+\frac{1}{6}+\cdots$, the sum for the first few terms in the series $((i.e) 1+1=2, 1+1+\frac{1}{2}=2.5, 1+1+\frac{1}{2}+\frac{1}{6}=2.6667$, …), the partial sums of first few terms in the series converges nearer to the value of e, equal to 2.71828 correct to three decimal places. Sometimes in an infinite series the individual terms tends to zero and the sequence of partial sums for this series tends to infinity. Such series does not possess the sum. For example, consider $\sum_{k=1}^{\infty}\frac{1}{k} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$.

Infinite series: Let a_r be a function of r which has a definite value for all integral values of r. An expression of the form $a_1+a_2+a_3+\cdots+a_n+\cdots$ in which every term is followed by another term, is called an infinite series. This series is denoted by $\sum_{r=1}^{\infty} a_r$.

Convergent series: An infinite series that has a finite sum is called convergent series. Otherwise it is divergent.

Partial sums and convergence: The sum of the infinite series represented by S and is defined by $S=\sum_{r=1}^{\infty} b_r$. Then the partial sums S_n of the series is defined by $S_n=\sum_{k=0}^{n} b_k$. If S_n has a limit, then the series converges. If the sequence of partial sums does not have a real limit, the series does not converge.

Taylor series: Consider the Taylor series $f(x)=\sum_{n=0}^{\infty} a_n\,(x-x_0)^n$, the coefficient a_n is obtained by repeated differentiation which is evaluated at $x=x_0$.

(i.e) $a_n = \frac{1}{n!}\frac{d^n}{dx^n} g(x_0)$

Taylor series expansion is valid only for functions which are continuous and differentiable.

Fourier Series: Fourier series is an infinite series representation of periodic function in terms of trigonometric sine and cosine functions. Most of the single valued functions which occur in applied mathematics can be expressed in the form of Fourier series which is in terms of sines and cosines. It is a very powerful method to solve ordinary and partial differential equations, particularly with periodic functions appearing as non- homogeneous terms.

Let f(x) be a well defined, periodic and piecewise continuous on [-l, l]. Then the

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Fourier series of f(x) is the infinite series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

A different approach is followed to calculate the Fourier series coefficients called orthogonality integrals. By simple elementary techniques, we define the coefficients that

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$ and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Fourier series is possible not only for continuous functions but also for periodic functions, functions which are discontinuous in their values and derivatives. Further, because of the periodic nature, Fourier constructed for one period is valid for all values.

Bessel's inequality:

If $|f|^2$ is integrable, then $\sum_{-N}^{N}|f(x)|^2 \le \frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2dt$ is known as Bessel's inequality.

When N→∞, the above equation becomes Parseval's formula.

Riemann Zeta function: The most interesting and fascinating Riemann Zeta function deals with the series of powers of natural numbers $\sum_{n=1}^{\infty}\frac{1}{n^2}$. The function exactly defined for real arguments as ζ(s) = $\sum_{n=1}^{\infty}\frac{1}{n^s}$ for s > 1.Moreover viewing ζ(s) as a function of complex variable than a real variable extends the function onto the entire complex plane $\mathfrak{C}$ except greater than unity.

Consider the expression n^s where s = x + iy.

Now, $n^s = e^{s\log n}$

But $e^x = \sum_{n=0}^{\infty}\frac{x^k}{k!}$ for $z \in \mathfrak{C}$.

Consequently $n^s = \sum_{n=0}^{\infty}\frac{(s\log n)^k}{k!} = \sum_{n=0}^{\infty}\frac{s^k(\log n)^k}{k!}$

Expressing the Riemann zeta function interms of prime numbers as

$\zeta(s) = \prod_p\left(1-\frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty}$ where p is a prime number and the product $\prod_p$ is taken for all primes.

The series $\sum_{n=1}^{\infty}$ and the Euler product $\prod_p\left(1-\frac{1}{p^s}\right)$ converges uniformly in the half plane s>1.

II. SOLUTION OF $\sum_{n=1}^{\infty}\frac{1}{n^2}$ BY USING PARTIAL SUMS

Consider the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$

If we add up the first two terms, we get

$$\frac{1}{1^2}+\frac{1}{2^2} = \frac{5}{4} = 1.25$$

If we add up the first three terms, we get

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2} = \frac{49}{36} = 1.3611$$

Similarly, by adding the consecutive terms, we get

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2} = \frac{205}{144} = 1.4236$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2} = \frac{2747}{1800} = 1.52611$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2} = 1.5538$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2} = 1.5743$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2} = 1.584$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^2} = 1.602$$

and so on.

The above partial sums form another infinite sequence $\left\{1, \frac{5}{4}, \frac{49}{36}, \frac{205}{144}, \frac{2747}{1800}, \ldots\right.$. The partial sums of this sequence approximately converges to $1.644 \approx \frac{\pi^2}{6}$

III. SOLUTION OF $\sum_{n=1}^{\infty}\frac{1}{n^2}$ BY USING FOURIER SERIES:

We remember the Fourier series of $(\pi - x)^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left\{\frac{4}{n^2}\cos nx\right\}$ in $(0,2\pi)$ is

$$(\pi - x)^2 = \frac{\pi^2}{3} + 4\left\{\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right\}$$

Setting x = 0 gives,

$$f(x) = (\pi - x)^2 = \frac{(\pi - 0)^2 + (\pi - 2\pi)^2}{2} = \frac{\pi^2 + \pi^2}{2} = \pi^2$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left\{\frac{1}{1^2}\cos 0 + \frac{1}{2^2}\cos 2(0) + \frac{1}{3^2}\cos 3(0) + \cdots\right\}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right\}$$

$$\frac{2\pi^2}{3} = 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right\}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

IV. SOLUTION OF $\sum_{n=1}^{\infty}\frac{1}{n^2}$ BY USING RIEMANN ZETA FUNCTION:

Consider $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

By Weierstrass factorization theorem,

$$\frac{\sin x}{x} = \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1+\frac{x}{3\pi}\right)\left(1-\frac{x}{3\pi}\right)\ldots$$

$$= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots$$

Multiply out this product and collect all the x^2 terms and by induction that the x^2 coefficient of $\frac{\sin x}{x}$ is

$$-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots\right)=-\frac{1}{\pi^2}$$

But the coefficient of x^2 is $-\frac{1}{3!}=-\frac{1}{6}$ in the infinite series expansion. These two coefficients must be equal.

$$\therefore -\frac{1}{6}=-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \therefore \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

V. RESULTS

In this paper we find many ways of convergence and its relationship. We have many methods to evaluate infinite series..

VI. CONCLUSION

The study of infinite series has wide range of application in science and engineering. The solution of the infinite series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ by different methods have been determined.

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