\( \lambda - \text{Fuzzy Subgroup} \)

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**Abstract**: As an extension to Rosenfeld’s definition of a fuzzy subgroup, a new kind of subgroup called \( \lambda \)-fuzzy subgroup of a group is defined and properties are studied. The image and preimage of a \( \lambda \)-fuzzy subgroup under homomorphism are also studied.

**Keywords**: Fuzzy Subgroup; \( \alpha - \text{Fuzzy Subgroup} \); \( \lambda - \text{Fuzzy Subgroup} \); Homomorphism.

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**INTRODUCTION**

The concept of fuzzy set was introduced by Zadeh [7] in 1965. Rosenfeld [4] introduced the notion of a fuzzy subgroup of a group \( G \). Since then, many authors gave generalization to the concept [1], [5], [6]. In [5], Sharma defined a \( \alpha - \) fuzzy subgroup as follows.

A fuzzy set \( A \) in a group \( G \) is called a \( \alpha - \) fuzzy subgroup of \( G \) if

\[ A^\alpha(xy) \geq A^\alpha(x) \land A^\alpha(y) \]

\[ A^\alpha(x^{-1}) = A^\alpha(x) \quad \forall x, y \in G \quad \text{and} \quad \alpha \in [0,1] \]

where \( A^\alpha = A(x) \Delta \alpha \).

But then any fuzzy set \( A \) in a group \( G \) is a \( \alpha = 0 \) fuzzy subgroup of \( G \), which we think to be modified. So we introduce another kind of fuzzy subgroup called \( \lambda - \) fuzzy subgroup.

The other motivation include to generalize Rosenfeld’s fuzzy subgroup, with minimum number of appearance of the parameter (if any). The second condition in the definition of a \( \lambda - \) fuzzy subgroup resulted from the thought that whenever \( A(x) \geq \lambda \), implies \( A(x^{-1}) \geq \lambda \). Similarly \( A(x) \leq \lambda \) implies \( A(x^{-1}) \leq \lambda \).

**II. PRELIMINARIES**

We review some basic definitions and results.

**Definition 2.1** [7] A fuzzy set \( A \) in a set \( X \) is a function \( A : X \rightarrow [0,1] \).

**Definition 2.2** [2] If \( A \) is a fuzzy set in a set \( X \), then for any \( t \in [0,1] \), the set \( A_t = \{ x \in X : A(x) \geq t \} \) is called a level subset of \( A \).

**Definition 2.3** [7] If \( A \) and \( B \) are two fuzzy sets in a set \( X \), then their intersection \( A \cap B \) is defined as

\[ (A \cap B)(x) = A(x) \land B(x), \quad \forall x \in X. \]

**Definition 2.4** [7] Let \( f \) be a function defined on a set \( X \). Let \( A \) be a fuzzy set in \( X \) and \( B \) be a fuzzy set in \( f(X) \). Then the image of \( A \) under \( f \) is defined by

\[ f(A)(y) = \max_{x \in f^{-1}(y)} A(x) \quad \forall y \in f(X). \]

If \( f \) is bijective, then \( f(A)(y) = A(x) \).

The preimage of \( B \) under \( f \) is defined by,

\[ f^{-1}(B)(x) = B(f(x)) \quad \forall x \in X. \]

**Definition 2.5** [4] A fuzzy set \( A \) in a set \( X \) has the Sup property if, for any subset \( T \subseteq X \), there exists \( t_0 \in T \) such that

\[ A(t_0) = \sup_{t \in T} A(t). \]

**Definition 2.6** [4] A fuzzy set \( A \) in a group \( G \) is a fuzzy subgroup of \( G \) if

\[ (i) \ A(xy) \geq A(x) \land A(y) \quad \text{and} \quad (ii) \ A(x^{-1}) = A(x), \quad \forall x, y \in G. \]

**Definition 2.7** [5] Let \( A \) be a fuzzy set in a group \( G \) and \( \alpha \in [0,1] \). Then \( A \) is called a \( \alpha - \) fuzzy subgroup of \( G \) if \( A^\alpha(xy^{-1}) \geq A^\alpha(x) \land A^\alpha(y) \quad \forall x, y \in G \), where \( = A(x) \Delta \alpha \).

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**A. Final Stage**

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III. \(\lambda\) - Fuzzy Subgroup and its Properties

Here, for an abstract group \(G\), we define a \(\lambda\) - fuzzy subgroup and list some of its properties.

**Definition 3.1** Let \(A\) be a fuzzy set in a groupoid \(G\) and \(\lambda \in (0,1]\). Then \(A\) is said to be a \(\lambda\) - fuzzy subgroupoid of \(G\) if

\[ A(xy) \geq A(x) \land A(y) \land \lambda \]

**Definition 3.2** Let \(A\) be a fuzzy set in a group \(G\) and \(\lambda \in (0,1]\). Then \(A\) is said to be a \(\lambda\) - fuzzy subgroup of \(G\) if

(i). \(A(xy) \geq A(x) \land A(y) \land \lambda \)

(ii). \(A(x) \land A(x^{-1}) \geq A(x) \land A(y) \land \lambda \) \(\forall x, y \in G\).

**Remark 3.1** We define the zero fuzzy set in a group \(G\) defined by \(0(x) = 0\), \(\forall x \in G\) to be the only 0 - fuzzy subgroup of \(G\). So, by a \(\lambda\) - fuzzy subgroup of a group \(G\), we mean a non-zero fuzzy set in \(G\).

**Remark 3.2** \(\lambda = 1\) leads to Rosenfeld’s fuzzy subgroup of \(G\), i.e., every Rosenfeld’s fuzzy subgroup is a \(\lambda\) - fuzzy subgroup. But the converse need not be true. For example: Let \(G = \{1, -1\}\) be a group under ordinary multiplication. A fuzzy subset \(A\) is defined as \(A(1) = 0.4, A(-1) = 0.6\). Then \(A\) is not a fuzzy subgroup of \(G\), for

\[(A \cap A) (xy) = A(xy) \land A(x^{-1}) \geq (A(x) \land A(y)) \land \lambda = (A \land A) (x) \land (A \land A) (y) \land \lambda.

Also,

\[(A \cap A) (x^{-1}) \land (A \cap A) (x) = (A \land A) (x^{-1}) \land (A \land A) (x) \]

\[= (A \land A) (x^{-1}) \land (A \land A) (x) \land (A \land A) (x) \land \lambda.

It follows that,

\[(A \land A) (x^{-1}) \land (A \land A) (x) \geq (A \land A) (x) \land (A \land A) (x) \land \lambda

Corollary 3.1. Intersection of a family of \(\lambda\) - fuzzy subgroups of a group \(G\) is again a \(\lambda\) - fuzzy subgroup of \(G\).

**Result:**

\[0.4 = A(1) < A(-1) \land A(-1) = 0.6.

Take \(\lambda = 0.4\), then

\[0.4 = A(1) \geq A(-1) \land A(-1) \land 0.4\]

It can also be verified that \(A(xy) \geq A(x) \land A(y) \land 0.4\) and \(A(x) \land A(x^{-1}) \geq A(x) \land 0.4\) holds \(\forall x, y \in G\).

**Theorem 3.1**. If \(A\) is a \(\lambda\) - fuzzy subgroupoid of a finite group \(G\), then \(A\) is a \(\lambda\) - fuzzy subgroup of \(G\).

**Proof.** Let \(x \in G\). Since \(G\) is finite, it is possible to find a positive integer \(n\) such that \(x^n = e\), where \(e\) is the identity in \(G\). Hence \(x^{-1} = x^{n-1}\).

Now,

\[A(x^n) = A(x^{n-1}) = A(x) \land A(x^{-1}) \geq A(x) \land A(y) \land \lambda = A(x) \land A(y) \land \lambda.

i.e., \(A(x^{-1}) \land A(x) \geq A(x) \land A(y) \land \lambda = A(x) \land A(y) \land \lambda\).

Thus, \(A\) is a \(\lambda\) - fuzzy subgroup of \(G\).

**Theorem 3.2**. Intersection of two \(\lambda\) - fuzzy subgroups of a group \(G\) is again a \(\lambda\) - fuzzy subgroup of \(G\).

**Proof.** Let \(A_1\) and \(A_2\) are two \(\lambda\) - fuzzy subgroups of a group \(G\).

**Remark 3.3**. The union of two \(\lambda\) - fuzzy subgroups of a group \(G\) may not be a \(\lambda\) - fuzzy subgroup of \(G\).

**Example:** Let \(G = \mathbb{Z}\), the set of integers under ordinary addition. Define two fuzzy sets in \(G\) by

\[A(x) = \begin{cases} 0.6, & \text{if } x \in 3\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}\]

\[B(x) = \begin{cases} 0.2, & \text{if } x \in 2\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}\]

Then \(A\) and \(B\) are \(\lambda\) - fuzzy subgroups of \(G\)

\(\forall \lambda \in (0,1]\), in particular \(\lambda = 0.5\).
Now, \((A \cup B)(x) = \begin{cases} 
0.6, & \text{if } x \in 3Z \\
0.2, & \text{if } x \in 2Z - 3Z \\
0, & \text{if } x \in 3Z, x \in 2Z
\end{cases}\)

Let \(x = 3, y = 8\). \((A \cup B)(3) = 0.6, (A \cup B)(8) = 0.2\), but, \((A \cup B)(x) < \Lambda (A \cup B)(3) \Lambda (A \cup B)(8) \Lambda 0.5\), i.e., \(A \cup B\) is not a \(\lambda\)-fuzzy subgroup of \(G\).

From now onwards \(e\) denotes the identity element in \(G\).

Theorem 3.3. Let \(A\) be a \(\lambda\) – fuzzy subgroup of a group \(G\). Then \(A(e) \neq 0\) for any non-zero fuzzy set \(A\) in \(G\).

Proof. For \(x \in G\),

\[A(e) = A(xx^{-1}) \geq A(x) \Lambda A(x^{-1}) \Lambda \lambda = A(x) \Lambda \lambda,\]

by condition (ii) in the Definition of a \(\lambda\) – fuzzy subgroup.

Also, if \(A \neq 0\), then \(A(x) > 0\), for some \(x \in G\). Hence \(A(e) \neq 0\).

Corollary 3.2. By the above theorem,

\[A(e) \Lambda \lambda \geq A(x) \Lambda \lambda, \quad \forall x \in G.\]

Theorem 3.4. If \(A\) is a \(\lambda\) – fuzzy subgroup of a group \(G\), then

\[A(xy^{-1}) = A(e) \Rightarrow A(x) \Lambda A(y) \Lambda \lambda \geq A(y) \Lambda \lambda, \quad \forall x, y \in G.\]

Proof. If \(A\) is a \(\lambda\) – fuzzy subgroup of \(G\), for all \(x, y \in G\),

\[A(x) = A(xy^{-1})y \geq A(xy^{-1}) \Lambda A(y) \Lambda \lambda = A(e) \Lambda A(y) \Lambda \lambda \geq A(y) \Lambda \lambda, \quad \forall x, y \in G.\]

by theorem 3.3.

Corollary 3.3. If \(A\) is a \(\lambda\) – fuzzy subgroup of a group \(G\),

\[A(xy^{-1}) = A(e) \Rightarrow A(x) \Lambda A(y) \Lambda \lambda \geq A(y) \Lambda \lambda, \quad \forall x \in G.\]

Proof. For a \(\lambda\) – fuzzy subgroup \(A\) of \(G\),

\[A(y) \geq A(y) \Lambda A(x) = A(y) \Lambda A(xy^{-1}) = A(xy^{-1}) \Lambda A(x) \Lambda \lambda \geq A(xy^{-1}) \Lambda A(x) \Lambda \lambda \Lambda \lambda \geq A(xy^{-1}) \Lambda A(x) \Lambda \lambda \Lambda \lambda\]

Hence, \(A(y) \Lambda \lambda \geq A(x) \Lambda \lambda\).

By the above theorem, \(A(x) \Lambda \lambda \geq A(y) \Lambda \lambda\). It follows that \(A(x) \Lambda \lambda = A(y) \Lambda \lambda\).

Theorem 3.5. Let \(A\) be a fuzzy set in a group \(G\) and let \(\mathbf{p} = \inf \{A(x) : x \in G\} \neq 0\). Then \(A\) is a \(\lambda\) – fuzzy subgroup of \(G\) for all \(\lambda \leq p\).

Proof. For \(x, y \in G\),

\[A(x) \Lambda A(y) \geq A(xy) \Lambda \lambda \]

\[\forall x, y \in G.\]

Also, since \(A(x^{-1}) \Lambda \lambda = \lambda, \quad A(x) \Lambda A(x^{-1}) \geq A(x^{-1}) \Lambda \lambda = \lambda\)

Hence, \(A\) is a \(\lambda\) – fuzzy subgroup of \(G\).

Theorem 3.6. Let \(t \in [0, 1]\), \(A(e) \geq t\) and \(A\) be a \(\lambda\) – fuzzy subgroup of \(G\). Then the level subset \(A_t\) is a subgroup of \(G\).

Proof. Clearly, \(A\) is non empty. Let \(x, y \in A_t\), then

\[A(x) \geq t, A(y) \geq t.\]

Since \(A\) is a \(\lambda\) – fuzzy subgroup of \(G\),

\[A(xy) \geq A(x) \Lambda A(y) \Lambda \lambda \geq t.\]

This means \(xy^{-1} \in A_t\). Therefore, \(A\) is a \(\lambda\) – fuzzy subgroup of \(G\).

Theorem 3.7. Let \(G\) be a group and \(A\) be a fuzzy set in \(G\) such that \(A_t\) is a subgroup of \(G\) for all \(t \in [0, 1]\).

Proof. Let \(x, y \in G\) and let \(A(x) = t_1\) and \(A(y) = t_2\). Then \(x \in A_{t_1}, y \in A_{t_2}\). Let us assume that

\[A(xy^{-1}) = A(e) \subset A_{t_1} \cap A_{t_2}.\]

Thus \(x, y \in A_{t_1} \cap A_{t_2}\). Hence, \(xy \in A_{t_1} \cap A_{t_2}\). Therefore, \(A(xy) \geq t_1 = A(x) \Lambda A(y) \Lambda \lambda \).

Also, \(x \in G\) and \(A(x) = t\). Then \(x \in A_t\).
Since $A_t$ is subgroup of $G$, $x^{-1} \in A_t$, which implies $A(x^{-1}) \geq t$.

Now $A(x) \land A(x^{-1}) \geq A(x) \land \lambda = t$ where $\lambda = k$.

Thus $A$ is a $\lambda = k$ - fuzzy subgroup of $G$.

Theorem 3.8. A fuzzy set $A$ in a group $G$ is a $\lambda$ - fuzzy subgroup of $G$ if and only if

$A(xy)^{-1} \geq A(x) \land A(y) \land \lambda$.

Proof. Suppose $A$ is a $\lambda$ - fuzzy subgroup of $G$. Then

$A(xy)^{-1} = A(xy^{-1}y^{-1}) \geq A(xy^{-1}) \land A(y^{-1}) \land \lambda \geq A(xy^{-1}) \land A(y) \land A(y^{-1}) \land \lambda \geq A(x) \land A(y^{-1}) \land A(y) \land \lambda \geq A(x) \land A(y) \land \lambda \land \lambda$,

by the repeated application of conditions (i) and (ii) of a $\lambda$ - fuzzy subgroup.

Conversely, let $A(xy^{-1}) \geq A(x) \land A(y) \land \lambda$, for a fuzzy set $A$ in $G$.

Then $A(xy) = A(x(y^{-1})^{-1}) \geq A(x) \land A(y^{-1}) \land \lambda$.

Therefore,

$A(xy) \land A(y) \geq A(x) \land A(y^{-1}) \land \lambda \land A(y) \geq A(x) \land A(y) \land \lambda$.

by condition (ii) of a $\lambda$ - fuzzy subgroup.

But then, $A(xy) \geq A(x) \land A(y) \geq A(x) \land A(y) \land \lambda$.

Also,$A(y^{-1}) = A(y)^{-1} = A(e) \land A(y) \land \lambda \geq A(y) \land \lambda$,

using theorem 3.3.

Hence, $A(y^{-1}) \land A(y) \geq A(y) \land A(y) \land \lambda$.

Theorem 3.9. Let $A$ be the characteristic function of a non-empty subset $H$ of a group $G$. Then $A$ is a $\lambda$ - fuzzy subgroup of $G$ if and only if $H$ is a subgroup of $G$.

Proof. Clearly, $A$ is a fuzzy set in $G$. First, let $A$ be a $\lambda$ - fuzzy subgroup of $G$.

For $x, y \in H, A(x) = A(y) = 1$.

Now, $A(xy) \geq A(x) \land A(y) \land \lambda = A(xy) = 1$. Thus, $xy \in H$.

Also,$A(x^{-1}) \land A(x^{-1}) \geq A(x) \land \lambda = A(x^{-1}) = 1 \Rightarrow x^{-1} \in H$.

Therefore, $H$ is a subgroup of $G$.

Conversely, if $H$ is a subgroup of $G$, then its characteristic function is fuzzy subgroup of $G$ and hence is a $\lambda$ - fuzzy subgroup of $G$.

IV. HOMOMORPHISM AND $\lambda$ - FUZZY SUBGROUPS

Theorem 4.1. A homomorphic preimage of a $\lambda$ - fuzzy subgroup of a group $G$ is a $\lambda$ - fuzzy subgroup of $G$.

Proof. Let $f : G_1 \rightarrow G_2$ be a group homomorphism. Let $B$ be a $\lambda$ - fuzzy subgroup of $G_2$.

For $x_y \in G_2$, $(f^{-1}(B))(xy) = B(f(xy)) = B(f(x)f(y)) \geq B(f(x)) \land B(f(y)) \land \lambda = (f^{-1}(B))(x) \land (f^{-1}(B))(y) \land \lambda$.

Also,(f^{-1}(B))(x^{-1}) \land (f^{-1}(B))(x) = b(f(x^{-1})) \land B(b(f(x))) = B(f(x^{-1}) \land B(f(x)) \land \lambda.

Thus, $f^{-1}(B)$ is a $\lambda$ - fuzzy subgroup of $G_1$.

Theorem 4.2. Let $A$ be a $\lambda$ - fuzzy subgroup of a group $G$. If $f : G_1 \rightarrow G_2$ is a bijective homomorphism, then $f(A)$ is a $\lambda$ - fuzzy subgroup of the group $G_2$.

Proof. Let $f(x), f(y) \in G_2$. (since, $f(G_1) = G_2$). Let $f(A) = B$.

Then $B(f(x)f(y)) = B(f(xy)) = A(xy) \geq A(x) \land A(y) \land \lambda = B(f(x)) \land B(f(y)) \land \lambda$.

Also,$B(f(x^{-1})) \land B(f(x)) = B(f(x^{-1})) \land B(f(x)) = A(x^{-1}) \land A(x)$.

Hence, $A(x) \land A(y) = B(f(x)) \land \lambda$, proves that $A$ is a $\lambda$ - fuzzy subgroup of $G_2$.

Theorem 4.3. A homomorphic image of a $\lambda$ - fuzzy subgroup of a group $G$ which has the Sup property is a $\lambda$ - fuzzy subgroup of $G$.

Proof. Let $f : G_1 \rightarrow G_2$ be a group homomorphism and $A$ be a $\lambda$ - fuzzy subgroup of $G_2$, $f(A) = B$.

Let $f(x), f(y) \in f(G_1)$. Let $x_0 \in f^{-1}(f(x)), y_0 \in f^{-1}(f(y))$ such that

$A(x_0) = \sup_{t \in f^{-1}(f(x))} A(t), \quad A(y_0) = \sup_{t \in f^{-1}(f(y))} A(t)$.

Now, since $x_0 y_0 \in f^{-1}(f(xy))$,
\[ B(f(x)f(y)) = \sup_{z \in f^{-1}(f(x)f(y))} A(z) = \]
\[ \sup_{z \in f^{-1}(f(x)f(y))} A(z) \geq A(x_0y_0) \geq A(x_0) \wedge A(y_0) \wedge \lambda = B(f(x)) \wedge B(f(y)) \wedge \lambda. \]

Also,
\[ B(f(x)^{-1}) \wedge B(f(x)) = \]
\[ \sup_{t \in f^{-1}(f(x)^{-1})} A(z) \wedge \sup_{t \in f^{-1}(f(x))} A(z) \geq A(x_0^{-1}) \wedge A(x_0) \geq A(x_0) \wedge \lambda = B(f(x)) \wedge \lambda, \text{proves the theorem}. \]

V. CONCLUSION

In this paper we find some sets in fuzzy and its properties and some structures of that sets.

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