λ - Fuzzy Subgroup

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***Abstract:** As an extension to Rosenfeld's definition of a fuzzy subgroup, a new kind of subgroup called λ fuzzy subgroup of a group is defined and properties are studied. The image and preimage of a λ fuzzy subgroup under homomorphism are also studied.*

***Keywords:** Fuzzy Subgroup; $\alpha-$ Fuzzy Subgroup; $\lambda-$Fuzzy Subgroup; Homomorphism.*

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INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [7] in 1965. Rosenfeld [4] introduced the notion of a fuzzy subgroup of a group G. Since then, many authors gave generalization to the concept [1], [5], [6]. In [5], Sharma defined a $\alpha-$ fuzzy subgroup as follows.

A fuzzy set A in a group G is called a $\alpha-$fuzzy subgroup of G if

$A^\alpha(xy) \geq A^\alpha(x) \wedge A^\alpha(y)$ (ii)

$A^\alpha(x^{-1}) = A^\alpha(x) \quad \forall\, x, y \in G$ and $\alpha \in [0,1]$

where $A^\alpha = A(x) \wedge \alpha$.

But then any fuzzy set A in a group G is a $\alpha = 0$ fuzzy subgroup of G, which we think to be modified. So we introduce another kind of fuzzy subgroup called $\lambda-$fuzzy subgroup.

The other motivation include to generalize Rosenfeld's fuzzy subgroup, with minimum number of appearance of the parameter (if any). The second condition in the definition of a $\lambda-$ fuzzy subgroup resulted from the thought that whenever $A(x) \geq \lambda$, implies $A(x^{-1}) \geq \lambda$. Similarly $A(x) \leq \lambda$ implies $A(x^{-1}) \leq \lambda$.

II. PRELIMINARIES

We review some basic definitions and results.

Definition 2.1 [7] A fuzzy set A in a set X is a function $A : X \to [0,1]$.

Definition 2.2 [2] If A is a fuzzy set in a set X, then for any $t \in [0,1]$, the set $A_t = \{x \in X : A(x) \geq t\}$ is called a level subset of A.

Definition 2.3 [7] If A and B are two fuzzy sets in a set X, then their intersection $A \cap B$ is defined as

$(A \cap B)(x) = A(x) \wedge B(x), \quad \forall\, x \in X.$

Definition 2.4 [7] Let f be a function defined on a set X. Let A be a fuzzy set in X and B be a fuzzy set in $f(X)$. Then the image of A under f is defined by

$f(A)(y) = \sup_{x \in f^{-1}(y)} A(x) \quad \forall\, y \in f(X)$. If f is bijective, then $f(A)(y) = A(x)$.

The preimage of B under f is defined by,

$f^{-1}(B)(x) = B(f(x)) \quad \forall\, x \in X.$

Definition 2.5 [4] A fuzzy set A in a set X has the Sup property if, for any subset $T \subseteq X$, there exists $t_0 \in T$ such that $A(t_0) = \sup_{t \in T} A(t)$.

Definition 2.6 [4] A fuzzy set A in a Group G is a fuzzy subgroup of G if

(i) $A(xy) \geq A(x) \wedge A(y)$ and (ii) $A(x^{-1}) = A(x)$, $\forall\, x, y \in G$.

Definition 2.7 [5] Let A be a fuzzy set in a group G and $\alpha \in [0,1]$. Then A is called a $\alpha-$ fuzzy subgroup of G if $A^\alpha(xy^{-1}) \geq A^\alpha(x) \wedge A^\alpha(y); \quad \forall x, y \in G$ where $= A(x) \wedge \alpha$.

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III. λ – FUZZY SUBGROUP AND ITS PROPERTIES

Here, for an abstract group G, we define a $\lambda -$ fuzzy subgroup and list some of its properties.

Definition 3.1 Let A be a fuzzy set in a groupoid G and $\lambda \in (0,1]$. Then A is said to be a $\lambda -$ fuzzy subgroupoid of G if $A(xy) \geq A(x) \wedge A(y) \wedge \lambda$.

Definition 3.2 Let A be a fuzzy set in a group G and $\lambda \in (0,1]$. Then A is said to be a $\lambda -$ fuzzy subgroup of G if

(i). $A(xy) \geq A(x) \wedge A(y) \wedge \lambda$

(ii). $A(x) \wedge A(x^{-1}) \geq A(x) \wedge \lambda \quad \forall x, y \in G.$

Remark 3.1 We define the zero fuzzy set in a group G defined by $0(x) = 0, \ \forall x \in G$ to be the only $0 -$ fuzzy subgroup of G. So, by a $\lambda -$ fuzzy subgroup of a group G, we mean a non- zero fuzzy set in G.

Remark 3.2 $\lambda = 1$ leads to Rosenfeld's fuzzy subgroup of G. i.e, every Rosenfeld's fuzzy subgroup is a λ- fuzzy subgroup. But the converse need not be true. For Example: Let $G = \{1, -1\}$ be a group under ordinary multiplication. A fuzzy subset A is defined as $A(1) = 0.4, A(-1) = 0.6$. Then A is not a fuzzy subgroup of G, for

$0.4 = A(1) < A(-1) \wedge A(-1) = 0.6.$

Take $\lambda = 0.4$, then

$0.4 = A(1) \geq A(-1) \wedge A(-1) \wedge 0.4.$

It can also be verified that $A(xy) \geq A(x) \wedge A(y) \wedge 0.4$ and $A(x) \wedge A(x^{-1}) \geq A(x) \wedge 0.4$ holds $\forall x, y \in G.$

i.e, A is a $\lambda = 0.4 -$ fuzzy subgroup of G.

Result: $A(x^n) \geq A(x) \wedge \lambda$, where x^n denote the composite of x's under the operation in G.

Theorem 3.1. If A is a $\lambda -$ fuzzy subgroupoid of a finite group G, then A is a $\lambda -$ fuzzy subgroup of G.

Proof. Let $x \in G$. Since G is finite , it is possible to find a positive integer n such that $x^n = e$, where e is the identity in G. Hence $x^{-1} = x^{n-1}$.

Now,

$$A(x^{-1}) = A(x^{n-1}) = A(xx^{n-2}) \geq A(x) \wedge A(x^{n-2}) \wedge \lambda = A(x) \wedge \lambda$$

.

i.e, $A(x^{-1}) \wedge A(x) \geq A(x) \wedge \lambda \wedge A(x) = A(x) \wedge \lambda.$

Thus, A is a $\lambda -$ fuzzysubgroup of G.

Theorem 3.2. Intersection of two $\lambda -$ fuzzy subgroups of a group G is again a $\lambda -$ fuzzy subgroup of G.

Proof. Let A_1 and A_2 are two $\lambda -$ fuzzy subgroups of a group G.

$$(A_1 \cap A_2)(xy) = A_1(xy) \wedge A_2(xy) \geq (A_1(x) \wedge A_1(y) \wedge \lambda) \wedge (A_2(x) \wedge A_2(y) \wedge \lambda = (A_1(x) \wedge A_2(x)) \wedge (A_1(y) \wedge A_2(y)) \wedge \lambda$$
$$= (A_1 \cap A_2)(x) \wedge (A_1 \cap A_2)(y) \wedge \lambda.$$

Also,

$$(A_1 \cap A_2)(x^{-1}) \wedge (A_1 \cap A_2)(x) = (A_1(x^{-1}) \wedge A_2(x^{-1})) \wedge ((A_1(x) \wedge A_2(x))$$

$$= \left(A_1(x^{-1}) \wedge A_1(x)\right) \wedge \left(A_2(x^{-1}) \wedge A_2(x)\right) \geq (A_1(x) \wedge \lambda) \wedge (A_2(x) \wedge \lambda) = (A_1(x) \wedge A_2(x)) \wedge \lambda$$
$$= (A_1 \cap A_2)(x) \wedge \lambda.$$

It follows that,

$$(A_1 \cap A_2)(x^{-1}) \wedge (A_1 \cap A_2)(x) \geq (A_1 \cap A_2)(x) \wedge \lambda.$$

Corollary 3.1. Intersection of a family of $\lambda -$ fuzzy subgroups of a group G is again a $\lambda -$ fuzzy subgroup of G.

Remark 3.3. The union of two $\lambda -$ fuzzy subgroups of a group G may not be a $\lambda -$ fuzzy subgroup of G.

Example: Let $G = Z$, the set of integers under ordinary addition. Define two fuzzy sets in G by

$$A(x) = \begin{cases} 0.6, & \text{if } x \in 3Z \\ 0, & \text{otherwise} \end{cases}$$

$$B(x) = \begin{cases} 0.2, & \text{if } x \in 2Z \\ 0, & \text{otherwise} \end{cases}$$

Then A and B are $\lambda -$ fuzzy subgroups of G $\forall \lambda \in (0,1]$, in particular $\lambda = 0.5$.

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Now, $(A \cup B)(x) = \begin{cases} 0.6, & \text{if } x \in 3Z \\ 0.2, & \text{if } x \in 2Z - 3Z \\ 0, & \text{if } x \notin 3Z, x \notin 2Z \end{cases}$

Let $x = 3, y = 8$. $(A \cup B)(x) = 0.6$, $(A \cup B)(y) = 0.2$, but, $(A \cup B)(x + y) = (A \cup B)(11) = 0$ $< (A \cup B)(3) \wedge (A \cup B)(8) \wedge 0.5$. i.e, $A \cup B$ is not a $\lambda -$ fuzzy subgroup of G.

From now onwards e denotes the identity element in G.

Theorem 3.3. Let A be a $\lambda -$ fuzzy subgroup of a group G. Then $A(e) \geq A(x) \wedge \lambda$, $\forall x \in G$. Thus $A(e) \neq 0$ for any non- zero fuzzy set A in G.

Proof. For $x \in G$,

$A(e) = A(xx^{-1}) \geq A(x) \wedge A(x^{-1}) \wedge \lambda = A(x) \wedge \lambda$, by condition (ii) in the Definition of a $\lambda -$ fuzzy subgroup.

Also, if $A \neq 0$, then $A(x) > 0$, for some $x \in G$. Hence $A(e) \neq 0$.

Corollary 3.2. By the above theorem,

$A(e) \wedge \lambda \geq A(x) \wedge \lambda, \quad \forall x \in G.$

Theorem 3.4. If A is a $\lambda -$fuzzy subgroup of a group G, then

$A(xy^{-1}) = A(e) \Rightarrow A(x) \geq A(y) \wedge \lambda \quad \forall x, y \in G.$

Proof. If A is a $\lambda -$ fuzzy subgroup of G, for all $x, y \in G$,

$A(x) = A(xy^{-1}y) \geq A(xy^{-1}) \wedge A(y) \wedge \lambda =$
$A(e) \wedge A(y) \wedge \lambda \geq A(y) \wedge \lambda$,
by theorem 3.3.

Corollary 3.3. If A is a $\lambda -$ fuzzy subgroup of a group G,

$A(xy^{-1}) = A(e) \Rightarrow A(x) \wedge \lambda = A(y) \wedge \lambda \quad \forall x \in G.$

Proof. For a $\lambda -$ fuzzy subgroup A of G,

$A(y) \geq A(y) \wedge A(e) = A(y) \wedge A(xy^{-1}) = A(yx^{-1}x) \wedge A(xy^{-1}) \geq A(yx^{-1})$
$= A((xy^{-1})^{-1}) \wedge A(xy^{-1}) \wedge A(x) \wedge \lambda \geq A(xy^{-1}) \wedge A(x) \wedge \lambda = A(e) \wedge A(x) \wedge \lambda \geq A(x) \wedge \lambda.$

Hence, $A(y) \wedge \lambda \geq A(x) \wedge \lambda$.

By the above theorem, $A(x) \wedge \lambda \geq A(y) \wedge \lambda$. It follows that $A(x) \wedge \lambda = A(y) \wedge \lambda$.

Theorem 3.5. Let A be a fuzzy set in a group G and let $p = \inf\{A(x) : x \in G\} \neq 0$. Then A is a $\lambda -$ fuzzy subgroup of G for all $\lambda \leq p$.

Proof. Since $\lambda \leq p = \inf\{A(x) : x \in G\}$, $A(x) \geq \lambda$, $\forall x \in G$. Therefore, $A(xy) \geq \lambda$ and $A(x) \wedge A(y) \wedge \lambda = \lambda$. i.e, $A(xy) \geq A(x) \wedge A(y) \wedge \lambda$.

Also, since

$A(x^{-1}) \wedge \lambda = \lambda, \quad A(x) \wedge A(x^{-1}) \geq A(x^{-1}) \wedge \lambda = \lambda.$

Hence , A is a $\lambda -$ fuzzy subgroup of G.

Theorem 3.6. Let $t \in (0,1]$, $A(e) \geq t$ and A be a $\lambda \in [t, 1] -$ fuzzy subgroup of a group G. Then the level subset A_t is a subgroup of G.

Proof. Clearly, A is non empty. Let $x, y \in A_t$, then $A(x) \geq t, A(y) \geq t$.

Since A is a $\lambda -$ fuzzy subgroup of G, $A(xy) \geq A(x) \wedge A(y) \wedge \lambda \geq t$.

i.e, $A(xy) \geq t$.Hence $xy \in A_t$. Also, $x \in A_t$ implies $A(x) \geq t$. Since A is a $\lambda -$ fuzzy subgroup of G, $A(x) \wedge A(x^{-1}) \geq A(x) \wedge \lambda \geq t$, shows that $A(x^{-1}) \geq t$.

This means $x^{-1} \in A_t$. Therefore, A is a $\lambda -$ fuzzy subgroup of G.

Theorem 3.7. Let G be a group and A be a fuzzy set in G such that A_t is a subgroup of G $\forall t \in [0,1]$, $A(e) \geq t$, then A is a $\lambda = k -$ fuzzy subgroup of G, where

$k = \text{Sup}\{A(x) : x \in G\}$.

Proof. Let $x, y \in G$ and let $A(x) = t_1$ and $A(y) = t_2$. Then $x \in A_{t_1}, y \in A_{t_2}$. Let us assume that $t_2 < t_1$. Then $A_{t_2} \subseteq A_{t_1}$. So $y \in A_{t_1}$. Thus $x, y \in A_{t_1}$ and since A_{t_1} is a subgroup of G, $xy \in A_{t_1}$. Therefore,

$A(xy) \geq t_1 = A(x) \wedge A(y) \wedge \lambda$, where $\lambda = k$.

Also, let $x \in G$ and $A(x) = t$. Then $x \in A_t$.

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Since A_t is subgroup of G, $x^{-1} \in A_t$, which implies $A(x^{-1}) \geq t$.

Now $A(x) \wedge A(x^{-1}) \geq A(x) \wedge \lambda = t$, where $\lambda = k$.

Thus A is a $\lambda = k -$ fuzzy subgroup of G.

Theorem 3.8. A fuzzy set A in a group G is a $\lambda -$ fuzzy subgroup of G if and oly if $A(xy)^{-1}) \geq A(x) \wedge A(y) \wedge \lambda$.

Proof. Suppose A is a $\lambda -$ fuzzy subgroup of G.Then

$$A(xy^{-1}) = A(xy^{-1}yy^{-1}) \geq A(xy^{-1}) \wedge A(yy^{-1}) \wedge \lambda \geq A(xy^{-1}) \wedge A(y) \wedge A(y^{-1}) \wedge \lambda \geq A(xy^{-1}) \wedge A(y) \wedge \lambda \geq A(x) \wedge A(y^{-1}) \wedge A(y) \wedge \lambda \geq A(x) \wedge A(y) \wedge \lambda.$$

by the repeated application of conditions (i) and(ii) of a $\lambda -$ fuzzy subgroup.

Conversely, let $A(xy^{-1}) \geq A(x) \wedge A(y) \wedge \lambda$ for a fuzzy set A in G.

Then $A(xy) = A(x(y^{-1})^{-1}) \geq A(x) \wedge A(y^{-1}) \wedge \lambda$

Therefore,

$$A(xy) \wedge A(y) \geq A(x) \wedge A(y^{-1}) \wedge \lambda \wedge A(y) \geq A(x) \wedge A(y) \wedge \lambda.$$

by condition (ii) of a $\lambda -$ fuzzy subgroup.

But then, $A(xy) \geq A(xy) \wedge A(y) \geq A(x) \wedge A(y) \wedge \lambda$.

Also,

$A(y^{-1}) = A(ey^{-1}) = A(e) \wedge A(y) \wedge \lambda \geq A(y) \wedge \lambda$,

using theorem 3.3.

Hence, $A(y^{-1}) \wedge A(y) \geq A(y) \wedge A(y) \wedge \lambda = A(y) \wedge \lambda$.

Theorem 3.9. Let A be the characteristic function of a non-empty subset H of a group G. Then A is a $\lambda -$ fuzzy subgroup of G if and only if H is a subgroup of G.

Proof. Clearly, A is a fuzzy set in G. First, let A be a $\lambda -$ fuzzy subgroup of G.

For $x, y \in H$, $A(x) = A(y) = 1$.

Now, $A(xy) \geq A(x) \wedge A(y) \wedge \lambda \Rightarrow A(xy) = 1$. Thus, $xy \in H$.

Also,

$A(x) \wedge A(x^{-1}) \geq A(x) \wedge \lambda \Rightarrow A(x^{-1}) = 1 \quad \Rightarrow x^{-1} \in H$.

Therefore, H is a subgroup of G.

Conversely, if H is a subgroup of G, then its characteristic function is fuzzy subgroup of G and hence is a $\lambda -$ fuzzy subgroup of G.

IV. Homomorphism and λ – Fuzzy Subgroups

Theorem 4.1. A homomorphic preimage of a $\lambda -$ fuzzy subgroup of a group G is a $\lambda -$ fuzzy subgroup of G.

Proof. Let $f : G_1 \to G_2$ be a group homomorphism. Let B be a $\lambda -$ fuzzy subgroup of G_2.

For $x, y \in G_1$, $(f^{-1}(B))(xy) = B(f(xy)) = B(f(x)f(y)) \geq B(f(x)) \wedge B(f(y)) \wedge \lambda = (f^{-1}(B))(x) \wedge (f^{-1}(B))(y) \wedge \lambda$.

Also,$(f^{-1}(B))(x^{-1})) \wedge (f^{-1}(B))(x) = B(f(x^{-1})) \wedge B(f(x)) = B(f(x)^{-1}) \wedge B(f(x))$

$\geq B(f(x)) \wedge \lambda = (f^{-1}(B))(x) \wedge \lambda$.

Thus, $f^{-1}(B)$ is a $\lambda -$ fuzzy subgroup of G_1.

Theorem 4.2. Let A be a $\lambda -$ fuzzy subgroup of a group G_1. If $f : G_1 \to G_2$ is a bijective homomorphism, then $f(A)$ is a $\lambda -$ fuzzy subgroup of the group G_2.

Proof. Let $f(x), f(y) \in G_2$. (since, $f(G_1) = G_2$.) Let $f(A) = B$.

Then

$$B(f(x)f(y)) = B(f(xy)) = A(xy) \geq A(x) \wedge A(y) \wedge \lambda = B(f(x)) \wedge B(f(y)) \wedge \lambda.$$

Also

$, B(f(x)^{-1}) \wedge B(f(x)) = B(f(x^{-1})) \wedge B(f(x)) = A(x^{-1}) \wedge A(x)$

$\geq A(x) \wedge \lambda = B(f(x)) \wedge \lambda$, proves that A is a $\lambda -$ fuzzy subgroup of G_2. □

Theorem 4.3. A homomorphic image of a $\lambda -$ fuzzy subgroup of a group G which has the Sup property is a $\lambda -$ fuzzy subgroup of G.

Proof. Let $f : G_1 \to G_2$ be a group homomorphism and A be a $\lambda -$ fuzzy subgroup of G_1, $f(A) = B$.

Let $f(x), f(y) \in f(G_1)$. Let $x_0 \in f^{-1}(f(x))$, $y_0 \in f^{-1}(f(y))$ such that

$$A(x_0) = \sup_{t \in f^{-1}(f(x))} A(t), \quad A(y_0) = \sup_{t \in f^{-1}(f(y)} A(t).$$

Now, since $x_0 y_0 \in f^{-1}(f(xy))$,

$$B\big(f(x)f(y)\big) = \sup_{z \in f^{-1}(f(x)f(y))} A(z) = \sup_{z \in f^{-1}(f(xy))} A(z) \geq A(x_0y_0) \geq A(x_0) \wedge A(y_0) \wedge \lambda = B(f(x)) \wedge B(f(y)) \wedge \lambda.$$

Also,

$$B\ (f(x)^{-1}) \wedge B\big(f(x)\big) = \sup_{t \in f^{-1}(f(x)^{-1})} A(z) \ \wedge \sup_{t \in f^{-1}(f(x))} A(z) \geq A(x_0^{-1}) \wedge A(x_0) \geq A(x_0)) \wedge \lambda = B(f(x)) \wedge \lambda,$$ proves the theorem.

V. CONCLUSION

In this paper we find some sets in fuzzy and its properties and some structures of that sets.

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