On The P Convexity of Graphs

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Abstract: *Let G be a finite, simple undirected graph and let S be a set of vertices of G. If every vertex having two neighbors inside S is also in S, then S is a P_3-convex set. The P_3-convex hull $H_c(S)$ of S is the smallest P_3 convex set containing S. If $H_c(S) = V(G)$, we say that S is a P_3-hull set of G. The cardinality $h_c(G)$ of a minimum P_3-hull set in G is called the P_3-hull number of G. In this paper we determine the P_3-hull number of some special graphs. Characterization of a tree is obtained in terms of convex sets. Study of convex invariants also done.*

Keywords: Convexity, Hull number.

I. INTRODUCTION

The spread disease on a square grid is an example of a problem in which the P_3-hull number of graphs can be directly applied. In square grid some cells are infected. Iteratively, an uninfected cell becomes infected if at least two of its neighbors are so. What is the minimum number of originally infected cells to guarantee that all cells of the grid become eventually infected? We approach the above problem through P_3-convexity [4]

A family $\mathcal{C}$ of subsets of a nonempty set X is called a convexity on X if

□∈ □, X ∈□

□ is stable for intersections, and

□ is stable for nested unions

(X,□) is called a convexity space and members of □ are called convex sets.

For a graph G with vertex set $V(G)$, a graph convexity on $V(G)$ is a collection □ of subsets of $V(G)$, to be regarded as convex sets such that

□∈ □, $V(G)$ ∈□.

Arbitrary intersections of convex sets are convex.

Every nested union of convex sets are convex.

The convex hull $H_c(S)$ of a set S of vertices of G is the smallest set in □ containing S. Several well known graph convexities are defined [6]. In this paper we study exclusively the P_3- convexity of graphs. The P_3 $-$convexity was first considered for directed graphs[2].

II. PRELIMINARIES

Definition 2.1: Let G be a graph. Given a set $S \subseteq V(G)$, the P_3- interval $I[S]$ of S is formed by S, together with every vertex outside S with at least two neighbors in S. If $I[S] = S$, then S is P_3- convex. A set $S \subseteq V(G)$ is P_3- convex, if every vertex having two neighbors inside S is also in S. The P_3-convex hull $H_c(S)$ of S is the smallest P_3 - convex set containing S.

The P_3- convex hull can be formed from a sequence $I_p[S]$, where p is a nonnegative integer, $I_0[S]$, =S, $I_1[S]$ =I[S], and $I_p[S]$=I[$I_{p-1}[S]$], for every p≥ 2. When, for some $p \in N$, we have $I_q[S]$= $I_p[S]$, for all q≥p, then $I_p[S]$, is a convex set. If $H_c(S) = V(G)$, then S is said to be a P_3-hull set of G. The cardinality $h_c(G)$ of a minimum P_3-hull set in G is called the P_3-hull number of G [4].

Definition 2.2:The Caratheodory number of G is the smallest integer c such that for every set S of vertices of G and vertex u in $H_c(S)$, there is a set $F \subseteq S$ with $|F| \leq c$ and $u \in H_c(S)$.

Definition 2.3: If R is a subset of G , then a Radon partition of R is a partition of R into two disjoint sets R_1 and R_2 with $H_c(R_1) \cap H_c(R_2) \neq \emptyset$. The Radon number $r(G)$ of G is the minimum integer r such that every set of r vertices of G has a Radon partition. [5]

Definition 2.4: A wounded spider is the graph formed by subdividing at most $n-1$ edges of the star $K_{1,n-1}$.

Definition 2.5: From a simple graph G, Mycielski's construction produces a simple graph G^1 containing G. Beginning with G having vertex set $\{v_1, v_2, \dots \dots \dots, v_n\}$, add vertices $\{u_1, u_2, \dots \dots \dots, u_n\}$ and one more vertex w. Add edges to make u_i adjacent all of $N_G(v_i)$, and finally let N(w)= $\{u_1, u_2, \dots \dots \dots, u_n\}$.

Definition2.6:Trestled graph of index k, $T_k(G)$ is the graph obtained from G by adding k copies of K_2 for each edge (u, v) of G and joining u and v to the respective end vertices of each K_2.

Definition 2.7: A convexity space X is said to be join hull commutative (JHC) if for any convex set C and $p \in X$, $Co(C \cup \{p\}) = \cup \{Co(\{c, p\}): c \in C\}$

III. CONVEX INVARIANTS

Theorem 3.1.$h_c(G) = n$ if and only if each vertex has degree less than or equal to 1.

Proof: If each vertex has degree less than or equal to 1. If S is a proper subset of $V(G)$ then $v \in V - S$ can not belong to P_3 - convex hull of S. Thus minimum P_3 $-$ convex hull set is V itself. And $h_c(G) = n$ .If $h_c(G) = n$, then no vertex is adjacent to more than two vertex. Each vertex is adjacent to atmost on vertex. Hence degree of each vertex is less than or equal to 1.

Theorem 3.2.$h_c(G) = n-1$ if and only if G is a star, where G is a connected graph with $n \geq 4$ vertices.

Proof: If G is a star $K_{1,n-1}$, then all the pendant vertices must be there in minimum P_3 $-$ convex hull set. Thus h_c ($K_{1,n-1}$)=n-1.

If $h_c(G) = n-1$, then there is only one vertex which

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belong to convex hull of other vertices, say v_n. No vertex except possibly v_n can be adjacent to v_l and v_m for $1 \leq l,m \leq n-1$. $(N[v_l]-\{v_n\}) \cap (N[v_m]-\{v_n\})=\square$, $1 \leq l,m \leq n-1$. Thus v_n is the only one vertex which is adjacent to v_i, $1 \leq i \leq n-1$. Since G is connected, $v_i 1 \leq i \leq n-1$ is adjacent to v_n. And thus G is a star.

Theorem 3.3. There exist a connected graph G with n vertices and $h_c(G) = a$, $2 \leq a \leq n-1$

Proof: When $a = n-1$, If $G=K_{1,n-1}$ then $h_c(G) = n-1$. For a ≤n-2, let G be the graph obtained from the star $K_{1,n-1}$ by joining vertices x_{i-1},x_i for $1 \leq i \leq n-a$, when $K_{1,n-1}$ is the star with centre x and pendant vertices $x_1, x_2,\ldots\ldots,x_{n-1}$. Then $x_{n-1}, x_{n-2},\ldots\ldots,x_{n-a}$ will be a minimum P_3 −convex hull set and hence $h_c(G) = a$.

Theorem 3.4.If G is disconnected with atleast two components G_1 and G_2, $n(G_1)\geq 2$ $n(G_2)\geq 2$. Then $h_c(\bar{G})=2$.

Proof:$K_{r,s}$r,s ≥2 is a spanning subgraph of $\bar{G}$. Also $h_c(K_{r,s})=2$ r,s≥2 and $h_c(K_{r,s})\geq h_c(\bar{G})$. But hull number of a graph is always greater than or equal to 2. Thus $h_c(\bar{G})=2$.

Theorem 3.5. $h_c(G) \leq \Delta+1$, when G is a wounded spider.

Proof: If $\{v_1, v_2,\ldots\ldots\ldots\ldots,v_n\}$ be the pendant vertices of the graph . Then $\{v_1, v_2,\ldots\ldots\ldots\ldots,v_n\}$ must be contained in a minimum P_3 − hull set. If atleast two vertices from this set is adjacent to the vertex with degree Δ. Then this set is a minimum P_3 -hull set and $h_c(G)=\Delta$. Otherwise, $\{v_1, v_2,\ldots\ldots\ldots\ldots,v_n\} \cup \{v\}$ where v is the vetex with degree Δ will form a minimum P_3 hull set.

Thus, $h_c(G)=\begin{cases} \Delta & \text{if centre is incident with more than one pendant vertex} \\ \Delta+1 & \text{otherwise} \end{cases}$.

Theorem 3.6. $h_c(T_k(G)) = km$ where m is the number of edges in G and $n \geq 3$ $k \geq 2$.

Proof: Let $u_{il}^p u_{jl}^p$ be the pth copy of K_2 added to the edge e_l with end vertices $v_i v_j$. Let $\{v_1, v_2,\ldots\ldots\ldots\ldots,v_{h_c(G)}\}$ be a minimum P_3 -hull set. Choose e_1 incident with v_1. Inductively choose e_i incident with v_i which is different from $e_1, e_2, \ldots\ldots e_{i-1}$ for 1≤i≤h(G). Take S= $\{u_{il}^p$ l: $1 \leq i,l \leq h_c(G)$ $1 \leq p \leq k\}$ Choose v_{i_l} a vertex incident with e_l for $h(G)+1 \leq l \leq m$. Then take T={ $u_{i_l l}{}^p 1 \leq p \leq k$ $h_c(G)+1 \leq l \leq m$ }.

Thus$|$ $S \cup T|$ $= km$ and we need atleast one vertex from each copy of K_2 to form the convex hull set. Thus $h_c(G) \geq km$. And these km vertices of $S \cup T$ will form a minimum convex hull set of $T_k(G)$.

Theorem 3.7. Let H be a connected subgraph of a graph G. Then if $V(H)$ is the only nontrivial convex set of H, then H is a block. In particular, if $V(G)$ is the only nontrivial convex set of G, then G is a block.

Proof: Suppose that $V(H)$ is the only nontrivial convex set of H. Then, if w is a cut vertex of H, then H_1 and H_2 are the components of $H-w$. Let v∈ H_1, then convex hull of $\{v,w\} \subseteq H_1$ which is a contradiction. Therefore, H is a block.

Theorem 3.8. Every block is a P_3 - convex set.

Proof: Let B be a block which is not P_3-convex set. Then, there exist a vertex u∈ $\bar{B}$ such that u has two neighbours v_1 and v_2 in B. $B \cup \{u\}$ is a non separablesubgraph containing B which is a contradiction to B is a block. Thus B is a P_3 -convex set.

Theorem 3.9. Let G be a connected graph with n vertices. The following statements are equivalent

1) $G$is a tree

2)There exist a sequence of sets $V(G) = V_p \supset V_{p-1} \supset \ldots\ldots\ldots \supset V_1$ where for each i, V_i is convex and $|V_i|$=i

3) For each connected subgraph$H$of G, $V(H)$ is a convex set of G.

Proof:-Let G be a tree Let v_1 be a pendant vertex of the tree G and V_{p-1}=V_p-$\{v_1\}$. Then clearly V_{p-1} is convex in G. Let v_2 be a pendant vertex of $G - v_1$ and V_{p-2}=V_{p-1}-$\{v_2\}$. Then clearly V_{p-2} is convex in G and so on . The sets V_i, thus formed have the property stated in 2.

Suppose that there exist a sequence of sets $V(G) = V_p \supset V_{p-1} \supset \ldots\ldots\ldots \supset V_1$ where for each i, V_i is convex and $|V_i|$=i. Let H be a connected subgraph of G. Then, if H is not convex there exist a point $u \in \overline{V(H)}$ such that u has two neighbours v_1 and $v_2 \in H$ then there are two paths in between u and v_1. Since G is a tree it is not possible. Therefore, $V(H)$ is a convex set of G.

Suppose that for each connected subgraph$H$of G, $V(H)$ is a convex set of G If G has a cycle 'C_n' then $C_n - v$ is a connected subgraph of G. But not convex . Thus G contains no cycle. Thus G is a tree.

Theorem 3.10. If G is a graph having no cycle with length ≤4 and n≥4, then $h_c(G)\geq 3$.

Proof: G is a graph, having the above property, then, if $S=\{u_1, u_2\} \subseteq V(G)$, then there exist atmost one vertex adjacent with both u_1 and u_2, sayu_3, thus $|I_1[S]|\leq 3$. Also, u_3 is adjacent to none of the vertices which is adjacent to $u_1$or u_2. Thus $H_c(S)=I_1[S] \neq V(G)$. Thus, $h_c(G)\geq 3$.

Theorem 3.11. If G is a graph with the property $I_1[S] = H_c(S)$, $\forall S \subseteq V(G)$, then G is a joint hull commutative P_3- convex graph. Converse is true for trees.

Proof; Let G be a graph with the property $I_1[S] = H_c(S)$, $\forall S \subseteq V(G)$, for any convex set C and $p \in V(G)$. If $w \in H_c(C \cup \{p\}) - (C \cup \{p\})$, then $w \in I^1(C \cup \{p\})$, then there exists $v_1 \in C$ such that w is adjacent to v_1 and p.

Then $w \in H_c(\{v_1,p\})$, $\forall v_1 \in C$.

And so, $H_c(C \cup \{p\}) = \cup \{H_c(\{c,p\}) : c \in C\}$

Thus, G is a joint hull commutative P_3- convex graph.

To prove the converse for trees, let T be a joint hull commutative P_3- convex tree ,

If $I_1[S] \neq H_c(S)$ for some $S \subseteq V(T)$

Then there exist vertex $v \in I_2[S]$ such that $v \notin I_1[S]$. Then there exist vertices $v_1, v_2 \in I_1[S]$ such that v is adjacent to v_1 and v_2 .Then we distinguish into two cases

Case 1.

If $v_1, v_2 \notin S$. Then there exist vertices $v_3, v_4, v_5, v_6 \in S$ such that v_1 is adjacent to v_3 andv_4 , v_2 is adjacent to v_5 andv_6. Since T is a tree

$\{v_3, v_4, v_5, v_6\}$ is an independent set.

If S=$\{v_3, v_4, v_5, v_6\}$and $p = v_1$, then S is a convex set. And $H_c(S\cup\{p\}) \subseteq \{v_1, v_2, v_3, v_4, v_5, v_6, v\}$. But $v \notin \cup \{H_c(\{v,p\}): v \in S\}$, which contradicts the fact that T is a joint hull commutative P_3- convex tree.

Case 2

If $v_1 \notin S$ and $v_2 \in S$. Then there exist vertices $v_3, v_4 \in S$ such that v_1 is adjacent to v_3 and v_4. Then $\{v_2, v_3, v_4\}$ is an independent set and S= $\{v_2, v_3\}$ is a convex set. Take p= v_4

Also $H_c(S\cup\{p\}) = \{v_1, v_2, v_3, v_4, v\}$. But $\cup \{H_c(\{v,p\}): v \in S\} == \{v_1, v_2, v_3, v_4\}$, which contradicts the fact that T is a joint hull commutative P_3- convex tree.

Thus ,$I_1[S] = H_c(S)$ for all $S \subseteq V(T)$.

Theorem 3.12. If G is a maximal outer planar graph with diam(G)>2, then G has joint hull commutative P_3- convex property , but $I_1[S] \neq H_c(S), S \subseteq V(G)$.

Proof: In maximal outer planar graph G convex sets are singleton sets or V(G) or vertex set S having property distance between any two vertices is atleast 3.

Let p$\in V(G)$

If S is either singleton or V(G) then trivially it satisfy the property$H_c(S\cup\{p\}) = \cup \{H_c(\{v,p\}): v \in S\}$.

If S is the vertex set having property distance between any two vertices is atleast 3,

If there is a vertex $v \in S$, such that d(v,p)=2, then $H_c(\{v,p\}) = V(G)$. And so, $H_c(S\cup\{p\}) = \cup \{H_c(\{v,p\}): v \in S\}$.

If there is no vertex $v \in S$, such that d(v,p)=2, then $H_c(\{v,p\}) = \{v,p\}$ for all v in S. And so $H_c(S\cup\{p\}) = S \cup \{p\} = \cup \{H_c(\{v,p\}): v \in S\}$.

Thus maximal outer planar graphs have joint hull commutative P_3- convex property.

Also, for any adjacent vertices u and w inV(G), if S={u,v}, $H_c(\{u,v\}) = V(G)$ and $I_1[S] \neq H_c(S)$.

Theorem 3.13. Let G^1be the graph generated by Mycielski's construction of a graph G with vertex set$\{v_1, \dots \dots v_n\}$. Then $h_c((G^1)) \leq h_c(G)+1$.

Let $\{v_1, v_2, \dots \dots \dots .., v_{h_c(G)}\}$ be a minimum P3- hull set of a graph G. Then If S= $\{v_1, v_2, \dots \dots \dots .., v_{h_c(G)}\} \cup \{v\}$, every vertex in $(V(G^1) - S)$ has atleast two twoneighbours in S. Thus $H_c(S) = V(G^1)$. Therefore, S is a P_3- hull set of (G^1). And $h_c(G^1) \leq h_c(G)+1$.

If $G = P_3$, then $h_c(G) = 2$. And if $G^1$is the graph generated by Mycielski's construction of a graph G, then $h_c(G^1) = 3$. Thus, this bound is sharp.

Remark 3.14. For a path P_n, c(P_n) =2.

Proof: Let $U \subseteq V(G)$, consider $v \in h(U)$. If $|U| \leq 2$or $v \in U$, there is nothing to prove. Otherwise, there exists two vertices v_1 and v_2 in U which are adjacent to v. Thus, if $F = \{v_1, v_2\}$, $v \in h(F)$. Hence c(P_n) =2.

Remark 3.15. For a complete graph K_n, c(K_n)=2.

Proof: Let $U \subseteq V(G)$, consider $v \in h(U)$. If $F = \{v_1, v_2\} \subseteq U$, then $H_c(F) = V(G)$. Thus, $v \in h(F)$. Hence c(K_n) =2.

Remark 3.16. For a cycle C_n , c(C_n)=2.

Proof: Let $U \subseteq V(G)$, consider $v \in h(U)$. If $v \in h(U) - U$, then $v \in I_1[S]$, because the degree of each vertex is 2 . Thus there exists two vertices v_1 and v_2 in U which are adjacent to v. Thus, if $F = \{v_1, v_2\}$, $v \in h(F)$. Hence c(C_n) =2.

Remark 3.17. For a star $K_{1,n}$, c($K_{1,n}$)=2.

Proof: Let $U \subseteq V(G)$, consider $v \in h(U)$. If $v \in h(U) - U$, then $v \in I_1[S]$, because the only one vertex having degree greater than 2 is the center . Thus there exists two vertices v_1 and v_2 in U which are adjacent to v. Thus, if $F = \{v_1, v_2\}$, $v \in h(F)$. Hence c($K_{1,n}$) =2.

Remark 3.18. For a star$K_{1,n}$, r($K_{1,n}$)=4, n≥3.

Proof: Every four element set can be partitioned into two sets having two elements for which the centre as the common element of convex hull. And for the set containing three vertices having degree one has no Radon partition. Thus r($K_{1,n}$)=4.

Remark 3.19. For a complete graph K_n, r(K_n)=3.

Proof: Convex hull of every set having atleast two elements contain all the other vertices. Thus every three element set can be partitioned into two sets for which the convex hull of one set contains the vertex from the other set. Hence r(K_n)=3.

Remark 3.20. For a complete bipartite graph $K_{m,n}$, r($K_{m,n}$)=3.

Proof: Every three element set contains atleast two vertices which are independent. And the convex hull of these vertices contain all the other vertices. Thus every three element set can be partitioned into two sets for which the convex hull of one set contains the vertex from the other set. Hence r($K_{m,n}$)=3.

IV. RESULTS

In this paper, we find P_3 – convex set. In this paper we determine the P_3 -hull number of some special graphs. Characterization of a tree is obtained in terms of convex sets. Study of convex invariants also done.

V. CONCLUSION

In this paper we conclude P_3-convex set, Now a days graph uses many networks and this paper how to reduce the minimized and reduce time also.

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