An Analysis of Discrete Time Geo/G/1 Retrial Queue with Second Optional Service with A Vacation

P. Gunasekaran, S.Jeyakumar

Abstract: In this paper we discussed a discrete time Geo/G/1 retrial queuing model with single vacation and second discretionary service. In this model the system gives primary vital service to every customers exhaustively and the customers appearing at the time of busy period becomes non persistent. After completion of this service only few customers ask for second optional service. Also, the system have idle of unsystematic distance and afterward this vacation the server starts its work only if customer available in the server otherwise the server is idle until minimum a single customer available in the server. In this model arrival times are geometrically distributed and service times and vacation times are commonly distributed. By creating function technique we obtain PGF and also we derived an analytical expression for mean queue length.

Keywords: Arrival and service time, Essential and optional service, Geo/G/1 retrial queue, Mean queue length, Non persistent customers.

I. INTRODUCTION

Due to applications of discrete queue in communication system and other related areas many researchers increase their interest in discrete queuing models. On the discrete queue basis a lot of computers and communication systems are modelled because in these computers and communication systems many events can happen only at regularly spaced intervals. Also, discrete queue is most suitable for Telecommunication system and computer modeling than continuous time queue.

In plenty of real life situations retrial queuing models has wide applications. Particularly, in telephone exchange, telecom networks, computer and communication systems. This queue has special characteristics is that, customer does not leave from the queue in the server while the system is serving another customer and enters in retrial group called orbit which the customer can do recurrent attempts to gain the service. Many survey research articles bibliographic art published on retrial queues [1] to [4].In the last few years only on the continuous case retrial queues were focused Madan [5], Medhi [6] and choudhury [7] have studies M/G/1 queue in which the service provided in two phases, but Yang and Li [8] have broaden this research to discrete retrial queue recently.

Atencia and Moreno [9] has studied elaborately about the mentioned queue with additional option for service deprived of system failures. Atencia, Moreno and Artoleju [10] discussed discrete bulk retrial queues with control of admission.

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M.Takahashi, H.Oswa and T.Fujisawa [11] also discussed regarding distinct time bulk reattempt queues with non-preemptive priority. Moreno [12] analyzed discrete reattempt queue with unreliable system and common system long lasting time. Gautam Choudhery [13] discuss M/G/1 queue with two stage service under D policy. Ramanath and kalidass [14] analyzed M/G/1 retrial queue with second optional service at various vacation policies and non persistent customers. Wang and Zhao [15] has analyzed discrete time reattempt queue with early downs and additional option for service.

We combine single vacation with above these concepts and we have developed a new concept namely, A discrete reattempt queue with additional option for service and single vacation and which has wide presentations in most of real world things, which is motivated me to develop this paper.

The aim of this article is to discuss the problem like that arises in customer call center. When customers makes call to a call center is first connected to receptionist, who gathers all the datas from the customer and response all the enquiries (first essential service). In this call center the customer may be allowed to contact technical person if this customer has some technical problems (second optional service). Otherwise he may be happy with the response from the receptionist. When call center is busy the customer may either decide to make repeated calls (attempts) to get response or switch off phone for the time being. After responding all calls the receptionist may allowed take rest (vacation) or may do some other works in the call center. This process is most suitable for our model under consideration.

II. MODEL DESCRIPTION

We study a discrete time retrial single server queue, which has the time axis that is separated into equally spaced time in terims known as slots. The important point is to specify the order where the departure and arrival happens, since in discrete queue the probability of entry and exist additional queuing events happening simultaneously might not be null. In literature we deals with pair of policies(i) if an arrival occurs first, then it is called as a late arrival system (LAS) or arrival first (AF) policy and (ii) if a departure occurs first, then it is called an early arrival system (EAS) or departure first (DF) policy. In this paper we follow case (ii), ie we study the type only for EAS. For accurate calculation, suppose that the customer departure take place in the time interval (m^-, m) and the customer arrival and retrials take place in the time interval (m, m^+), that is the customer entry, and the retrials happen sright after instantly the opening limits and the exist immediately before the slot boundary.

With probability p customer arrival process follows Bernoulli process. At busy period the customer decided to exist the server (1-r) (probability) hence he becomes non persistent and with probability r he joins the orbit (customer in the orbit are waiting to retry services).An arriving customer turn on the service station when the system is idle. Here the busy system probability is θ and idle system probability is $\overline{\theta}$.Once the primary service has done the customer can either receive additional optional service i.e. α probability or can be left from the server with complementary $\overline{\alpha}$ probability.

In the circle every client makes a free reattempt source and the reattempt time (the time between two progressive attempts by a similar client) obeys a mathematical distribution with probability 1 – r. For repeated customer the retrial process confirms only if, upon a specific trial, the system has no job service and the recurring client is chosen for service among all other recurring clients who are endeavoring the service around then and service station is actuated effectively.

Also, the system has break with probability η if there have been no client in the system. After returning from vacation if minimum one client available in the server the system starts its service with probability $\overline{\eta} = 1 - \eta$ otherwise the system is idle.

The service times $\{t_{1,k}\}$ and $\{t_{2,k}\}$respectively of first primary and second optional services are autonomous and generally distributed with probability generating function $T_1(x) = \sum_{k=1}^{\infty} t_{1k} x^k$ and $T_2(x) = \sum_{k=1}^{\infty} t_{2k} x^k$ respectively.

The nth factorial moments will be denoted by $\beta_1^{(n)}$ and $\beta_2^{(n)}$.respectively.

At last it is suppose that the vacation times, service times, retrial times and inter arrival times are mutually independent. Further, we suppose $0 \pi p \pi 1$, $0 \pi r \pi 1$,$0 \pi \theta \pi 1 and 0 \pi \eta \pi 1$ in order to avoid trivial cases We use the notation to denote by $\overline{p} = 1 - p$ and $\rho = \rho_1 + \rho_2$ the traffic intensity, where $\rho_1 = p\beta_{1,1}$ and $\rho_2 = \alpha p \beta_{2,1}$ respectively represent the server load because of arrival in the primary and secondary optional services.

III. THE MARKOV CHAIN

The state of the system at time n $^+$ is mentioned below

$$S_n = \{X_n, \zeta_n, C_n\}$$

where X_n represents the system state ie, if the service is unoccupied then X_n =0, if the system is busy with primary service and additional service then X_n =1and X_n =2 respectively and the server is on vacation then X_n =3 and C_n, the number of clients in the retrial group. If $X_n = (1,2)$, then $\zeta_1^{(n)}$ denotes the outstanding service time of the customer which is currently in service .If $X_n = 3$ $\zeta_2^{(n)}$ represents remaining vacation time.

Subsequent to presenting the above supplementary factors of S_n the future elements of depends just on the present state. Else on the other hand, given the present express, the following state, and the advancement of the framework preceding the present state are autonomous. So it can be demonstrated that [S_n , n € C } is the Markov chain of our queuing system, where the state space is S = $\{(0,k); k \geq 0, (j.i.k); j = 1,2,3, i \geq 1, k \geq 0\}$

IV. QUEUE LENGTH DISTRIBUTION

The second optional service served to some of the customers those who are completed their first essential service exhaustively only

The limiting probabilities are defined as

$$\pi_{0,j} = \lim_{n\to\infty} \Pr\{X_n = 0, N_n = j\}$$

$$\pi_{1,k,j} = \lim_{n\to\infty} \Pr\{X_n = 1, \xi_1^{(n)} = k, N_n = j\}$$

$$\pi_{2,k,j} = \lim_{n\to\infty} \Pr\{X_n = 2, \xi_1^{(n)} = k, N_n = j\}$$

$$\pi_{3,k,j} = \lim_{n\to\infty} \Pr\{X_n = 3, \xi_2^{(n)} = k, N_n = j\}$$

The Kolmogorov equations for the stationary distribution is

$$\pi_{0,j} = \overline{p} r^k \pi_{o,j} + \overline{\alpha}\,\overline{p} r^k \pi_{1,1,j} + \overline{p} r^k \pi_{2,1,j} + (1 - \delta_{0,k}) \overline{p} \pi_{3,1,j} \quad j \geq 0$$

,

$$\pi_{1,k,j} = p\theta t_{1,i}\pi_{0,j} + \overline{p}(1 - r^{j+1})\theta t_{1k}\pi_{0\,j+1} + \overline{\alpha} p\theta t t_{1k}\pi_{21j} + \overline{\alpha}\,\overline{p}(1 - r^{j+1})\theta t_{1k}\,\pi_{1,1,j+1} + (1 - \delta_{0,j}) p\pi_{1k+1j-1} ,+ p\theta t_{1k}\,\pi_{2,1,j}$$

$$+ \overline{p}\pi_{1,k+1,j} + \overline{p}(1 - r^{j+1})\theta t t_{1k}\pi_{2,1,j+1} + (1 - \delta_{0,j}) p\theta t_{1k}\pi_{3,1,j} + (1 - r^{j+1})\overline{p\theta} t_{1k}\pi_{3,1,j+1} \;, k \geq 1, j \geq 0 \qquad (2)$$

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$$\pi_{2,k,j} = (1-\delta_{0,j})p\alpha t_{2k}\pi_{1,1,j-1} + \alpha t_{2,k}\bar{p}\pi_{1,1,j} + (1-\delta_{0,j})p\pi_{2,k+1,j-1} + \bar{p}\pi_{2,k+1,j} + (1-\delta_{0,j})p\alpha t_{2,k}\pi_{3,1,j-1} + \bar{p}\alpha t_{2,k}\pi_{3,1j}$$

$k \geq 1,\ j \geq 0$ (3)

$$\pi_{3,k,j} = \bar{p}\eta v_k \pi_{1,1,j+1}\bar{\alpha}(1-r^{j+1}) + p\eta v_k \pi_{1,1,j}\ \bar{\alpha} + \bar{p}\eta v_k \pi_{2,1,j_{+1}}\bar{\alpha}(1-r^{j+1}) + p\eta v_k \pi_{2,1,j} + \bar{p}\pi_{3,k+1,j} + p\pi_{3,k+1,j-1}(1-\delta_{0,j})$$

$$+\bar{p}\eta v_k \pi_{j_{+1}}(1-r^{j+1}) + p\eta v_k \pi_{0,j} + \bar{p}\eta v_k \pi_{3,1,j+1}(1-r^{j+11}) + p\eta v_k \pi_{3,1,j} \qquad \mathrm{k} \geq 0, \mathrm{j} \geq 0 \qquad (4)$$

The normalization condition is'

$$\sum_{j=0}^{\infty}\pi_{0,j} + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\ \pi_{1,k,j} + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\pi_{2,k,j} + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\pi_{3,k,j} = 1$$

We introduce following generating function to solve the above system of equations

$$\Phi_0(z) = \sum_{j=0}^{\infty}\pi_{0,j}z^j \ \Phi_1(x,z) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\ \pi_{1,k,j}x^k z^j$$

$$\Phi_2(x,z) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\ \pi_{2,k,j}x^k z^j \ \Phi_3(x,z) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\ \pi_{3,k,j}x^k z^j$$

and also we define auxiliary generating function as follows

$$\Phi_{1,k}(z) = \sum_{j=0}^{\infty}\pi_{1,k,j}z^j \ \Phi_{2,k}(z) = \sum_{j=0}^{\infty}\pi_{2,k,j}z^j \ \Phi_{3,k}(z) = \sum_{j=0}^{\infty}\pi_{3,k,j}z^j$$

Multiply both sides by z ^{j} and taking summation over j in equation (1), we get

$$\Phi_0\ (z) = \bar{p}\Phi_0(rz) + \bar{p}\bar{\alpha}\Phi_{1,1}(rz) + \bar{p}\Phi_{2,1}(rz) + \bar{p}\Phi_{3,1}(rz) \qquad (5)$$

Multiply both sides by z ^{j} and taking summation over j in equation (2), we get

$$\Phi_{1,k}(z) = (\tau)\Phi_{1,k+1}(z) + \frac{(\tau)}{z}\theta t_{1,k}\left[\Phi_0(z) + \bar{\alpha}\Phi_{1,1}(z) + \Phi_{2,1}(z) + \Phi_{31}(z)\right] - \frac{\bar{p}}{z}\theta t_{1,k}\left[\Phi_0(rz) + \bar{\alpha}\Phi_{1,1}(rz) + \Phi_{2,1}(rz)\right]$$

$$-\frac{\bar{p}}{z}\theta t_{1,k}\left[\Phi_0(rz) + \bar{\alpha}\Phi_{1,1}(rz) + \Phi_{2,1}(rz) + \Phi_{3,1}(rz)\right] \text{where } \tau = \bar{p} + pz \ \ (6)$$

Multiply both sides by z ^{j} and taking summation over j in equation (3), we get

$$\Phi_{2,k}(z) = (\tau)\Phi_{2,k+1}(z) + (\tau)\alpha t_{2,k}\Phi_{1,1}(z) + (\tau)\alpha t_{2,k}\Phi_{3,1}(z) \qquad (7)$$

Multiply both sides by z ^{j} and taking summation over j in equation (4), we get

$$\Phi_{3,k}(z) = \frac{\eta v_k(\tau)}{z}\left[\phi_0(z) + \bar{\alpha}\phi_{11}(z) + \varphi_{21}(z) + \varphi_{31}(z)\right] + (\tau)\varphi_{3k+1}(z) - \frac{p\eta v_k}{z}\varphi_0(z) \qquad (8)$$

Using (5) in (6) and after some algebraic simplification, we obtain

$$\Phi_{1,k}(z) = (\tau)\Phi_{1,k+1}(z) - \frac{1-z}{z}\theta t_{1,k}\Phi_0(z)p + \frac{\tau}{z}\theta t_{1,k}\left[\bar{\alpha}\Phi_{1,1}(z) + \Phi_{21}(z) + \Phi_{3,1}(z)\right] \ \ (9)$$

Using (5) in (8) and doing some simplification, we obtain

$$\Phi_{3,k}(z) = (\tau)\Phi_{3,k+1}(z) - \frac{1-z}{z}\eta v_k\Phi_0(z) + \frac{\tau}{z}\eta v_k\left[\bar{\alpha}\Phi_{11}(z) + \Phi_{21}\ \ (z) + \Phi_{31}(z)\right] (10)$$

Multiply both sides by x^k and taking summation over k and doing some simplification in equation (9), we obtain

$$\Phi_1(x,z)\left[\frac{x-\tau}{x}\right] = \frac{\tau}{z}\left[\bar{\alpha}\theta T_1(x) - z\right]\Phi_{11}(z) - \frac{1-z}{z}\ p\Phi_0(z)T_1(x) + \frac{(\tau)}{z}\theta T_1(x)\left[\varphi_{21}(z) + \varphi_{31}(z)\right] \qquad (11)$$

Multiply both sides by x^k and taking summation over k and doing some simplification in equation (7), we obtain

$$\Phi_2(x,z)\left(\frac{x-\tau}{x}\right) = (\tau)\alpha\Phi_{1,1}(z)T_2(x) - (\tau)\Phi_{2,1}(z) + \alpha(\tau)T_2(x)\varphi_{31}(z) \ (12)$$

Multiply both sides by x^k and taking summation over k and doing some simplification in equation (10), we obtain

$$\Phi_3(x,z)\left(\frac{x-\tau}{x}\right) = \frac{\tau}{z}\Phi_{3,1}(z)\left[v(x) - z\right] - \frac{1-z}{z}\eta\varphi_0(z)v(x) + \frac{\tau}{z}\eta v(x)\left[\bar{\alpha}\varphi_{11}(z) + \varphi_{21}(z)\right]$$

(13)

In equation (11) putx = τ and solving for $\Phi_{1,1}(z)$, we get

$$\Phi_{1,1}(z)=\frac{T_1(x)\theta[(1-z)p\Phi_0(z)-\tau(\Phi_{2,1}(z)+\Phi_{3,1}(z)]}{\tau[\bar{\alpha}\theta T_1(\bar{p}+pz)-z]} \quad (14)$$

In equation (12) put x = $\bar{p}+pz$ and solving for $\Phi_{2,1}(z)$, we get

$$\Phi_{2,1}(z)=\frac{\tau\alpha(\Phi_{1,1}(z)+\Phi_{3,1}(z))T_2(x)}{\tau} \quad (15)$$

In equation (13) put x = $\bar{p}+pz$ and solving for $\Phi_{3,1}(z)$, we get

$$\Phi_{3,1}(z)=\frac{\eta v(x)[(1-z)\phi_0(z)-\tau(\bar{\alpha}\phi_{11}(z)+\phi_{31}(z))]}{[v(x)-z]\tau} \quad (16)$$

Using (16) in (15) and solving for $\Phi_{2,1}(z)$

$$\Phi_{2,1}(z)=\frac{\tau\alpha T_2(x)\varphi_{11}(z)(v(x)-z)+\alpha T_2(x)v(x)\varphi_0(z)[(1-z)-\bar{\alpha}\tau]}{\tau\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}} \quad (17)$$

Using (17) in (16) and solving for $\Phi_{3,1}(z)$

$$\Phi_{3,1}(z)=\frac{\eta v(x)[\varphi_{11}(z)[(1-z)-\bar{\alpha}\tau]]\{[[v(x)-z]+\alpha T_2(x)\eta v(x)]-\alpha T_2(x)v(x)\}}{[[v(x)-z]+\alpha T_2(x)\eta v(x)][v(x)-z]\tau} \quad (18)$$

Using (17) in (18) in (14) and solving for $\Phi_{1,1}(z)$, we get

$$\Phi_{1,1}(z)=\frac{\begin{array}{c}T_1(x)p\theta(1-z)[[v(x)-z]+\alpha T_2(x)]+v(x)[(1-z)-\bar{\alpha}\tau](1-\alpha T_2(x))\\ \{[[v(x)-z]+\alpha T_2(x)\eta v(x)]-\alpha zT_2(x)\}\tau(\bar{\alpha}\theta T_1(x)-z)\end{array}}{\tau\theta T_1(x)[\bar{\alpha}+\alpha T_2(x)-z]}\Phi_0(z) \quad (19)$$

Using (19) in (17) and solving for $\Phi_{2,1}(z)$, we get

$$\Phi_{2,1}(z)=\frac{\begin{array}{c}\alpha T_2(x)\varphi_0(z)[T_1(x)p\theta(1-z)[(v(x)-z)+\alpha T_2(x)\eta v(x)]]+(\bar{\alpha}\theta T_1(x)-z)(1-\alpha T_2(x))\}\\ \{v(x)[(1-z)-\bar{\alpha}\tau]\{[(v(x)-z)+\alpha T_2(x)\eta v(x)]\tau+1\}\end{array}}{[(v(x)-z)+\alpha T_2(x)\eta v(x)]\tau(\bar{\alpha}\theta T_1(x)-z)(1-\alpha T_2(x))} \quad (20)$$

Using (19) in (18) and solving for $\Phi_{3,1}(z)$, we get

$$\Phi_{3,1}(z)=\frac{[(1-z)-\bar{\alpha}\tau]\eta v(x)\varphi_0(z)}{[v(x)-z]}\left[\frac{1+\alpha T_2(x)v(x)\tau}{\tau}\right]$$

$$+\frac{\alpha T_2(x)\eta v(x)T_1(x)p\theta(1-z)\{[v(x)-z]+\alpha T_2(x)\}\varphi_0(z)}{\{[v(x)-z]+\alpha T_2(xz)\eta v(x)\}\tau[\bar{\alpha}\varphi T_1(x)-z][v(x)-z][1-\alpha T_2(x)]}$$

$$-\frac{\alpha T_2(x)v(x)[(1-z)-\bar{\alpha}(\tau)]\varphi_0(z)}{\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}[v(x)-z]}\left[\frac{v(x)}{\tau}+\alpha T_2(x)\eta v(x)\right]$$

Now, $\varphi_0[z]=\bar{p}\varphi_0[rz]+\bar{p}\varphi_{11}[rz]\bar{\alpha}+\bar{p}\varphi_{21}[rz]+\bar{p}\varphi_{31}[rz]$

$\varphi_0(z)=\bar{p}\Phi_0(rz)G(rz)$

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Where G(rz)= [1+A+B+C]

The expression A,B and C are given below

$$A = \frac{T_1\left(\bar{p}+prz\right)p\theta(1-rz)\left\{\left[v\left(\bar{p}+prz\right)-rz\right]\left[v\left(\bar{p}+prz\right)-rz\right]+\alpha T_2\left(\bar{p}+prz\right)v\left(\bar{p}+prz\right)\left(\bar{\alpha}+\eta\right)\right]\right\} + T_1\left(\bar{p}+prz\right)p\theta(1-rz)\left\{\alpha T_2\left(\bar{p}+prz\right)\left[2\left[v\left(\bar{p}+prz\right)-rz\right]+\alpha T_2\left(\bar{p}+prz\right)v\left(\bar{p}+prz\right)(2+\eta)\right]\right\}}{\left(\bar{p}+prz\right)\left(\bar{\alpha}\theta T_1\left(\bar{p}+prz\right)-rz\right)\left(1-\alpha T_2\left(\bar{p}+prz\right)\right)}$$

$$B = \frac{v\left(\bar{p}+prz\right)\left[(1-rz)-\bar{\alpha}\left(\bar{p}+prz\right)\right]}{\left(\bar{p}+prz\right)\left[v\left(\bar{p}+prz\right)-rz\right]\left\{\left[v\left(\bar{p}+prz\right)-rz\right]+\alpha T_2\left(\bar{p}+prz\right)\eta v\left(\bar{p}+prz\right)\right\}}$$

In (12) substitute the values of $\varphi_{11}(z)$, $\varphi_{21}(z)$, $\varphi_{31}(z)$, $\varphi_0(z)$ and on simplification we get

$$\Phi_1(x,z) = \frac{x\left(\bar{\alpha}\theta T_1(x)-z\right)\varphi_0(z)}{z^2(x-\tau)}\left\{\frac{T_1(x)p\theta(1-z)\left[\left[v(x)-z\right]+\alpha T_2(x)\right] + \tau v\left(\bar{p}+pz\right)\left\{\left[v(x)-z\right]+\alpha T_2(x)\eta v(x)\right\}\left(\bar{\alpha}\theta T_1(x)-z\right)\left[1-\alpha T_2(x)\right]}{\left\{\left[v(x)-z\right]+\alpha T_2(x)\eta v(x)\right\}\left(\bar{\alpha}\theta T_1(x)-z\right)\left[1-\alpha T_2(x)\right]}\right\}$$

Use the values of $\varphi_{11}(z)$, $\varphi_{21}(z)$, $\varphi_{31}(z)$, $\varphi_0(z)$ in equation (11) and on simplification we obtain

$$\phi_2(x,z) = \frac{x}{(x-(\tau))}\varphi_0(z)\left[\frac{\alpha T_2(x)s_1(x)p\theta(1-z)\left\{\left[v(x)-z\right]+\alpha T_2(x)\right\}\left\{\left[v(x)-z\right]+2\alpha T_2(x)\eta v(x)\right\}}{\left[\bar{\alpha}\theta T_1(x)-z\right]\left\{\left[v(x)-z\right]+\alpha T_2(x)\eta v(x)\right\}\left[v(x)-z\right]\left[1-\alpha T_2(x)\right]}\right]$$
$$+\frac{x}{(x-(\tau))}\varphi_0(z)\left\{\frac{\alpha T_2(x)\left[(1-z)-\bar{\alpha}(\tau)\right]+\left\{\left[v(x)-z\right]+2\alpha T_2(x)\eta v(x)\right\}(\tau)-\alpha T_2(x)v(x)\left[1+\alpha T_2(x)v(x)\right]}{\left\{\left[v(x)-z\right]+\alpha T_2(x)\eta v(x)\right\}\left[v(x)-z\right]}\right\}$$

Use the values of $\varphi_{11}(z)$, $\varphi_{21}(z)$, $\varphi_{31}(z)$, $\varphi_0(z)$ in equation (13) and on simplification we obtain

$$\phi_3(x,z) = \frac{x}{z(x-(\tau))}\varphi_0[z]\left\{\frac{p\theta(1-z)T_1(x)\left\{\left[v(x)-z\right]+\alpha T_2(x)\right\}\left\{\alpha T_2(x)\eta v(x)\left[v(x)-z\right]\right\} + p\theta(1-z)T_1(x)\left\{\left[v(x)-z\right]+\alpha T_2(x)\right\}\left\{\left[v(x)-z\right]+\alpha T_2(x)\eta v(x)\right\}\eta v(z)\bar{\alpha} + \left\{\left[v(x)-z\right]+\alpha T_2(x)\eta v(x)\right\}\eta v(z)\alpha T_2(x)}{\left\{\left[v(x)-z\right]+\alpha T_2(x)\eta v(x)\right\}\left[v(x)-z\right](\tau)}\right\}$$
$$+\frac{x}{z(x-(\tau))}\varphi_0[z]\left\{\frac{\left[(1-z)-\bar{\alpha}(\tau)\right]v(x)\left\{\begin{matrix}\left[1+\alpha T_2(x)\eta v(x)(\tau)\left[v(x)-z\right]\eta v(z)\right] \\ +\bar{\alpha}\left\{\left[v(x)-z\right]+\alpha T_2(x)\eta v(x)\right\}+\alpha s_2(x)\left(\bar{\alpha}+1\right)\end{matrix}\right\}}{\left[v(x)-z\right](\tau)}\right\}$$
$$+\frac{x}{z(x-(\tau))}\varphi_0[z]\left\{\frac{\alpha s_2(x)\left\{\left[(1-z)-\bar{\alpha}(\varsigma)\right]v(x)-(\tau)^2\alpha T_2(x)\right\}\eta v(z)\left[v(x)-z\right] - \left[(1-z)-\bar{\alpha}(\tau)\right]v(x)\left[v(x)-z\right]\left[v(x)+\alpha T_2(x)\eta v(x)(\tau)\right]}{\left[v(x)-z\right](\tau)\left\{\left[v(x)-z\right]+\alpha T_2(x)\eta v(x)\right\}}\right\}$$

We obtain the probability generating function $\Phi(z)$ by

$$\Phi(z) = \phi_0(z)+\phi_1(1,z)+\phi_2(1,z)+\phi_3(1,z)$$

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$$p\theta(1-z)T_1(x)[[v(x)-z]+\alpha T_2(x)][[v(x)-z]+\alpha T_2(x)\eta v(x)]$$

$$\varphi(z)=\frac{\left\{\frac{1}{z}\left[\bar{\alpha}\theta-z+\alpha T_2(x)\right]+\alpha sT(x)\eta v(x)[1+(v(x)-z)]+\eta v(z)\left[\bar{\alpha}+\alpha T_2(x)\right]\right\}}{z[1-(\tau)]\left[\bar{\alpha}\theta T_1(x)-z\right][1-\alpha T_2(x)][[v(x)-z]+\alpha T_2(x)\eta v(x)]}\varphi_0[z]$$

$$+\frac{\varphi_0[z]\left\{\frac{\bar{\alpha}\theta-z}{z}[[v(x)-z]+\alpha T_2(x)\eta v(x)](\tau)v(x)[1-\alpha T_2(x)]\left[\bar{\alpha}\theta T_1(x)-z\right]\right\}}{z[1-(\tau)]\left[\bar{\alpha}\theta T_1(x)-z\right][1-\alpha T_2(x)]}$$

$$+\frac{\varphi_0[z]\{v(x)[v(x)-z][\alpha T_2(x)(\tau)[1+\alpha T_2(x)\eta v(x)(\tau)]+[1+\alpha T_2(x)\eta v(x)(\tau)][[v(x)-z]+\alpha T_2(x)\eta v(x)]]\}}{z[1-(\tau)][[v(x)-z]+\alpha T_2(x)\eta v(x)][v(x)-z]}$$

$$-\frac{\varphi_0[z]\alpha s_2(x)\left\{\begin{array}{l} s_1(x)(\tau)^2[v(x)-z]+(\tau)\left[(1-z)+\bar{\alpha}(\tau)\right][v(x)+\alpha T_2(x)\eta v(x)(\tau)5] \\ -z\left\{\begin{array}{l} T_1(x)p\theta(1-z)[[v(x)-z]+\alpha T_2(x)\eta v(x)]^2+v(x)\left[(1-z)+\bar{\alpha}(\tau)\right] \\ \left[[v(x)-z]+\alpha T_2(x)\eta v(x)\right]^2(\tau)\left[\bar{\alpha}\theta T_1(x)-z\right][1-\alpha T_2(x)] \\ +z(\tau)[[v(x)-z]+\alpha T_2(x)\eta v(x)]\left[\bar{\alpha}\theta T_1(x)-z\right][1-\alpha T_2(x)] \end{array}\right\}[v(x)-z] \\ -\left\{\begin{array}{l} [v(x)\left[(1-z)+\bar{\alpha}(\tau)\right]-(\tau)^2\alpha T_2(x)]\eta v(z)[v(x)-z] \\ -\left[v(x)\left[(1-z)+\bar{\alpha}(\tau)\right][v(x)-z]\right][v(x)+\alpha T_2(x)\eta v(x)(\tau)] \end{array}\right\} \end{array}\right\}}{z[1-(\tau)][[v(x)-z]+\alpha T_2(x)\eta v(x)][v(x)-z]}$$

V. STEADY STATE CONDITION

For the system under discussion the steady state is obtained from $\Phi(1)=1$, and is obviously less than one, which is given by

$$\Phi(1)=\phi_0(1)+\phi_1(1,1)+\phi_2(1,1)+\phi_3(1,1)=1$$

$$\varphi(1)=\frac{2\bar{\alpha}p\eta\left(\bar{\alpha}\theta-1\right)[pE(v)-1]}{\phi_0(1)\left(\bar{\alpha}\theta-1\right)\left[2\bar{\alpha}p\eta[pE(v)-1]+2\eta(D)+2\bar{\alpha}[pE(v)-1](E)+2\bar{\alpha}\eta(F)+2\bar{\alpha}(G)\right]+\phi_0{}'(1)[(H)]+\phi_0{}''(1)[(I)]}$$

Where D is given by

$$G=[E(v)p-1]\{\alpha p[3E(T_2)+2]+1+\alpha p\theta+\alpha\overline{\bar{\alpha}(\bar{\alpha}\theta-1)+2p\eta[E(v)+E(T_2)]+\alpha}\}+$$

$$E(T_2)\left\{\begin{array}{l}\bar{\alpha}p^2[1+2\alpha\eta]+(1+\bar{\alpha}p)(1+2\alpha p)+2p^2(\alpha\bar{\alpha}+\alpha^2\eta\theta)+\alpha p(1-\alpha)(\bar{\alpha}\theta-1)[2p+\alpha\eta+1]\\ -\alpha^2p(\bar{\alpha}\theta-1)[\alpha\eta p+2]-\alpha^2p\eta+E(s_2)[\alpha^2p\eta[1-(\bar{\alpha}\theta-1)]-\alpha p(\bar{\alpha}\theta-1)]+E(s_1)\alpha\bar{\alpha}p\eta\end{array}\right\}+$$

$$E(v)\{\bar{\alpha}p^2[1+2\alpha\eta]+p(1+\bar{\alpha}p)[1+2\alpha\eta]+\alpha^2p^2\eta\theta+\alpha\bar{\alpha}(\bar{\alpha}\theta-1)+E(T_2)[\alpha p^2][1+2\alpha\eta]+\alpha^2p^2\eta(2-\alpha)\}+$$

$$E(T_1)\alpha^2p\eta\theta+[\bar{\alpha}\theta E(T_1)p-1][\alpha\bar{\alpha}[\alpha\eta+2p^2E(s_2)+1+\alpha\eta p]]+p(1+\bar{\alpha}p)[1+2\alpha\eta]+$$

$$2\left\{\begin{array}{l}E(T_2(T_2-1))p^2\left[\begin{array}{l}\alpha\bar{\alpha}(1+2\alpha\eta)+(\bar{\alpha}\theta-1)[(1-\alpha)[\alpha\eta+(\bar{\alpha}\theta-1)]+\alpha\eta[\eta(1-2\alpha)-\alpha^2]]+(1+\alpha)+2\alpha p^2-\alpha\\ {[\bar{\alpha}\theta E(s_1)p-1]}\end{array}\right]+\\ E(v(v-1))p^2[(\bar{\alpha}\theta-1)(1-\alpha)(1+2\alpha\eta)+(1+2\alpha)]+E(T_1(T_1-1))p^2[\alpha^2(1-\alpha)^2\theta(1+\eta)]\end{array}\right\}$$

$-$

$$H=2\left\{\begin{array}{l}[E(v)p-1][2\alpha+\eta-\alpha\bar{\alpha}(\bar{\alpha}\theta-1)^2\eta+\bar{\alpha}^3(\bar{\alpha}\theta-1)(1+\alpha\eta)(1+\eta)+2\alpha\bar{\alpha}(\bar{\alpha}\theta-1)+\alpha\bar{\alpha}(\bar{\alpha}\theta-1)\eta p+\eta]\\ +E(T_2)P\left(\begin{array}{l}\bar{\alpha}+2\bar{\alpha}\eta\alpha+\alpha\bar{\alpha}(\bar{\alpha}\theta-1)(1+\eta)--\alpha^3\eta(\bar{\alpha}\theta-1)-(\bar{\alpha}\theta-1)(\overline{\alpha^2}+\alpha^2)-2\alpha+2\overline{\alpha^2}\alpha\eta p(\bar{\alpha}\theta-1)\\ -(1+\alpha)+\alpha\overline{\alpha^2}(\bar{\alpha}\theta-1)^2-\bar{\alpha}(\bar{\alpha}\theta-1)^2\alpha^2(1+\alpha\eta)+\alpha\bar{\alpha}^2(\bar{\alpha}\theta-1)\end{array}\right)\\ E(v)p\left[\begin{array}{l}\bar{\alpha}p(1+\alpha\eta)+\alpha^2p(1+\bar{\alpha}\alpha(\bar{\alpha}\theta-1))+\bar{\alpha}\alpha\eta p-1-\alpha p+2\overline{\alpha^2}(\bar{\alpha}\theta-1)+\alpha^2\bar{\alpha}^2(\bar{\alpha}\theta-1)(\alpha+(\bar{\alpha}\theta-1))\\ +(\bar{\alpha}\theta-1)\alpha^2+\bar{\alpha}^2\alpha\eta(\bar{\alpha}\theta-1)+2\alpha\bar{\alpha}(\bar{\alpha}\theta-1)\end{array}\right]\\ +[\bar{\alpha}\theta E(T_1)p-1][\alpha^2\bar{\alpha}\eta[1+\bar{\alpha}(\bar{\alpha}\theta-1)]]+\{\bar{\alpha}p+(1+\alpha)(1+\bar{\alpha}p)+\alpha p\eta(2\bar{\alpha}+\alpha\theta)+\\ +(\bar{\alpha}\theta-1)\bar{\alpha}(\alpha\eta p+\alpha+p)\}(\alpha+\bar{\alpha}\theta)+\alpha^2\bar{\alpha}(\bar{\alpha}\theta+1)\end{array}\right\}$$

amd $I=\{\bar{\alpha}(1+\alpha\eta)[1+\alpha(\bar{\alpha}\theta-1)]-(1+\alpha)\}$

VI. PERFORMANCE MEASURE

Mean queue length is obtained below. From the PGF we get mean queue length by differentiating this PGF with respect to z and then put z = 1

$$E(q)=\frac{2\bar{\alpha}p\eta(\bar{\alpha}\theta-1)[pE(v)-1]}{\begin{array}{l}\phi_0(1)(\bar{\alpha}\theta-1)[2\bar{\alpha}p\eta[pE(v)-1]+2\alpha\eta(D)+2\bar{\alpha}[pE(v)-1](E)+2\bar{\alpha}\eta(F)+2\bar{\alpha}(G)]\\ +\phi_0{}'(1)[(H)]+\phi_0{}''(1)[(I)]\end{array}}$$

Where the terms D, E, F, G, H and I are mentioned in the earlier section steady state condition.

VII. PARTICULAR CASE

When the vacation is zero, the PGF of first primary and additional option that is second optional our model are reduced respectively into

$$\phi_1(x,z)=\frac{T_1(x)-T_1(\bar{p}+pz)}{x-(\bar{p}+pz)}\frac{p\theta x(1-z)}{\theta T_1(\bar{p}+pz)[\bar{\alpha}+\alpha T_1(\bar{p}+pz)]-z}\phi_0(z)$$

$$\phi_2(x,z)=\frac{T_2(x)-T_2(\bar{p}+pz)}{x-(\bar{p}+pz)}\frac{p\theta x(1-z)T_2(\bar{p}+pz)}{\theta T_2(\bar{p}+pz)[\bar{\alpha}+\alpha T_2(\bar{p}+pz)]-z}\phi_0(z)$$

and PGF of the orbit of our model is reduced into

$$\Phi(z)=\phi_0(z)+\phi_1(1,z)+\phi_2(1,z)+\phi_3(1,z)$$

$$\Phi(z)=\frac{\theta(1-z)T_1(\bar{p}+pz)(\bar{\alpha}+\alpha T_2(\bar{p}+pz))}{\theta T_1(\bar{p}+pz)[\bar{\alpha}+\alpha T(\bar{p}+pz)]-z}\phi_0(z)$$

Which is PGF of discrete time Geo/G/1 retrial queue with second optional service Wang and Zhao (15)

VIII. NUMERICAL EXAMPLES

This section has numerical examples are briefly discussed in two different cases. In both of these two cases mean queue length is investigated in following manner

1. The result on mean queue length when the arrival rate is increases

2. The result on mean queue length when the service rate is increases

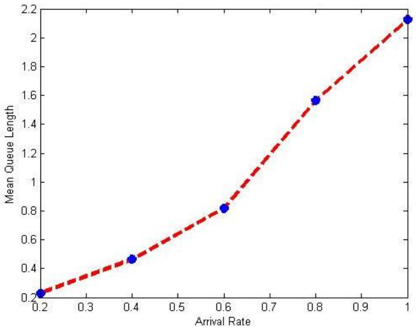
Case (I): In case (I), customer arrival times ,. Service times, vacation times are all geometrically distributed.

1. When the customer arrival rate rises the result on mean queue length is investigated below with the following index with graph.

Table 1.1

Arrival rate	$\varphi(1)$	E(Q)
.2	.18	.229
.4	.31	.464
.6	.46	.817
.8	.61	1.563
1.0	.84	2.124

Graph 1.1 Arrival rate Vs Mean queue length

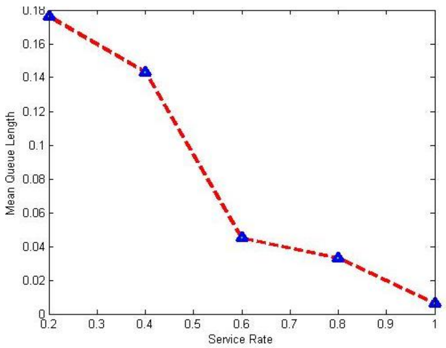


We notice that mean queue length increases when arrival rate rises which is inferred by above index and graph

2.When the server time rises the result on mean queue length is investigated below with the following index with graph.

Table 1.2

Service rate	$\varphi(1)$	Mean queue length
0.2	.71	.176
0.4	.59	.143
0.6	.41	.045
0.8	.34	.033
1.0	.21	.006



Graph 1.2 Service rate Vs Mean queue length

we notice that mean queue length decrease when service rate rises which is inferred by above index and table.

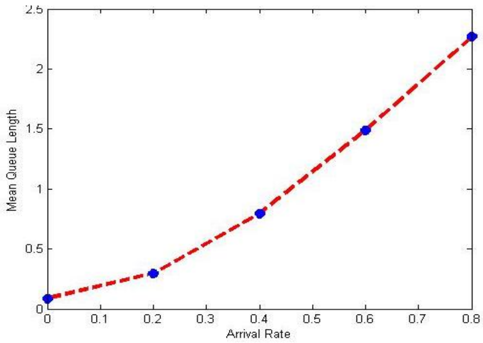
Case (II): In case (II) customer arrival times obey geometrical distribution and. Service times, vacation times are obeys Poisson distribution.

1When the customer arrival rate rises the result on mean queue length is investigated below with the following index with graph.

Table 2.1

Arrival rate	$\varphi(1)$	E(Q)
0	.31	0.087
.2	.47	0 .293
.4	.609	0 .795
.6	.74	1.49
.8	.87	2.27

Graph 2.1 Arrival rate Vs Mean queue length



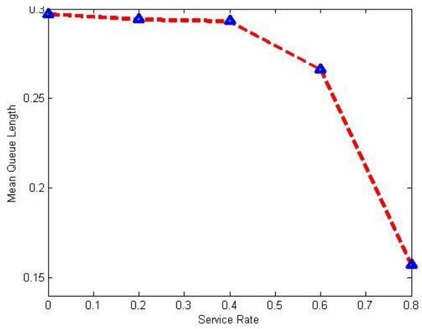
We notice that mean queue length increases when arrival rate rises which is inferred by above index and graph

2.When the server time rises the result on mean queue length is investigated below with the following index with graph.

Table 2.2

Service rate	$\varphi(1)$	E(Q)
0	.93	0.297
.2	.74	0.294
.4	.611	0.293
.6	.47	0.266
.8	.29	0.157

Graph 2.2 Service rate Vs Mean queue length



we notice that mean queue length decrease when service rate rises which is inferred by above index and table.

IX. CONCLUSION

Here we have analyzed the character of single vacation of discrete time Geo/G/1 retrial queue with non persistent client and additional option that is second optional service. In this model by using producing function technique the PGF have been derived. In performance measure an analytical expression for mean queue length has derived. The results of expected queue length are investigated in numerical examples with various arrival and service rates This model is most applicable in many real life situations.

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