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Minimum Total Irregularity of Totally Segregated ∞-Bicyclic Graph

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Abstract:*A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one. A bicyclic graph, in which any two adjacent vertices have distinct degrees, is a totally segregated bicyclic graph. Total Irregularity of a graph is defined as:* $irr_t(G)=\frac{1}{2}\sum_{u,v\in V(G)}|d_G(u)-d_G v|$*. In this paper, total irregularity of totally segregated ∞-bicyclic graph is discussed and some properties of totally segregated ∞- bicyclic graph G with Δ=4 and* $n_4(G)=1$ *is found. The basic bicycle denoted by∞(p,q,1) is obtained from two vertex-disjoint cycles CpandCq by identifying one vertex of Cp and one vertex of Cq. The ∞-bicyclic graph is obtained by attaching some trees to basic bicycle ∞(p,q,1). In this paper we determine the minimum total irregularity of totally segregated ∞- bicyclic graph on n vertices. Degree sequence of totally segregated ∞-bicyclic graph with minimum total irregularity is also found.*

Keywords: *Total irregularity, Totally segregated ∞-bicyclic graph, Basic bicycle ∞(p,q,1), Degree sequence.*

I. INTRODUCTION

In this paper we consider only simple undirected connected graphs. Let G=(V,E) be a graph of order n and size m. For u, v∈V(G), we denote the number of edges incident to v by $deg_G(v)$ or $d_G(v)$. The length of the shortest path from u to v by d(u, v). $N_G(v)$={w:d(v,w)=1} denotes neighborhood of v. For a graph G, let Δ(G) be the maximum degree of the vertices of G. Degree sequence of a graph is a non-increasing sequence of the vertex degrees of the graph.

One among them is the total irregularity of a graph introduced by Abdo, Brandt and Dimitrov [1], which is defined as

$irr_t(G)=\frac{1}{2}\sum_{u,v\in V(G)}|d_G(u)-d_G(v)|$. ……….(1)

Jackson and Entringer [9]considered the graphs in which any two adjacent vertices have distinct degrees, and these graphs are named as totally segregated. Jorry T. F. and ParvathyK. S. [10] studied a special case, by considering those trees in which degrees of any two adjacent vertices are differed by a constant k≠ 0, and these graphs are named as k-segregated tree. A graph G is k- segregated, if $|d_G(u)–d_G(v)|=k \neq 0$ for all edges uv∈ E(G) [10]. However, in many applications and problems, it is of great importance to know how irregular a given graph is. In [1] the graphs with maximal total irregularity are determined and the upper bound of the set $\{irr_t(G):|V(G)| = n\}$ is obtained as:

$irr_t(Gmax) = \frac{1}{12}(2n^3-3n^2-2n)$, if n even.

$= \frac{1}{12}(2n^3-3n^2-2n+3)$, if n odd.

In [1] authors also showed that, If T is a tree on n vertices, $irr_t(T) \leq (n-1)(n-2)$, and the star $K_{1,n-1}$ has the maximal total irregularity among all trees of same order. Star on n vertices $K_{1,n-1}$, n≥ 3 is totally segregated tree (TST), particularly it is (n-2)-segregated tree. Hence, maximum total irregularity of totally segregated trees on n vertices is (n-1)(n-2), and the maximum is attained for (n-2)-segregated tree i.e. for $K_{1,n-1}$. Characterization of totally segregated trees with Δ = 3 and minimum total irregularity of totally segregated trees on n vertices is found in [11].

Theorem [11]

Let Tbe a TST on n vertices and Δ(T)=3. Then $n_3(T)-1\leq n_2(T) \leq 2n_3(T)+1$.

In [2], they characterized the non-regular graphs with minimal total irregularity and thereby resolved the conjecture by Zhu, You and Yang [12] about the lower bound on the minimal total irregularity of non-regular connected graphs. Abdo and Dimitrov [2] also characterized the non-regular graphs with the second and the third smallest total irregularity.

Throughout this paper let n_i denote the number of vertices with degree i. Here, equation (1) is rewritten as:

$Irr_t(G) = \sum_{d_i>d_j}(d_i-d_j)(n_{d_i}-n_{d_j})$ ……………......(2)

where d_i,d_j are distinct degrees of vertices in G.Note that total irregularity of a given graph is completely determined by its degree sequence. Graphs with the same degree sequences have the same total irregularity.

In this paper we find minimum total irregularity of totally segregated ∞- bicyclic graph on n vertices. For the sake of convenience totally segregated bicyclic graph is called a TSB-graph. We represent degree sequence of graph in the following form($k^{n_k},(k-1)^{n_{k-1}},\cdots,3^{n_3},2^{n_2},1^{n_1}$).

II. TOTALLY SEGREGATED BICYCLIC(TSB) GRAPHS ON N VERTICES WITH MINIMUM TOTAL IRREGULARITY

In [13] authors introduce different classes of bicyclic graphs. A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one. The basic bicycle, denoted by ∞ (p,q,1), is obtained from two vertex-disjoint cycles Cp and Cq by identifying one vertex of Cp and one vertex of Cq, where p, q ≥3 . See Figure 1.

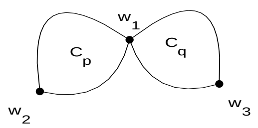

Figure 1

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A totally segregated bicyclic (TSB) graph is a bicyclic graph which is totally segregated.
See Figure 2.

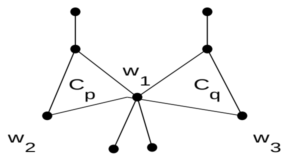

Figure 2

In this section, we determine minimum total irregularity of totally segregated ∞- bicyclic graph on n vertices.Observe that any ∞- bicyclic graphG is obtained from an basic bicycle∞- (p,q,1) by attaching trees to some of its vertices. If G is obtained from ∞ (p,q,1) by attaching trees to some of its vertices then we call G as bicyclic graph with basic bicycle ∞- (p,q,1). Obviously Bn is the set of those graphs each of which is a bicyclic graph with basic bicycle ∞- (p,q,1), (p≥ 3, q≥ 3) which is called ∞- bicyclic graph for convenience.

Remark 2.1

For n ≤ 6, a totally segregated ∞- bicyclic graph of order n does not exist.

Theorem 2.1

If G is a bicyclic graph with Δ(G)=k, then $n_1(G)= n_3(G)+2n_4(G)+\cdots+(k-2)n_k(G)-2$.

Proof. Let G be a bicyclic graph on nvertices.Thenn = $n_1+n_2+\cdots+n_k$. By first theorem of graph theory,2(n+1) = $n_1+2n_2+\cdots+kn_k$. Eliminating n and solving for n_1, we get
$n_1(G)= n_3(G)+2n_4(G)+3n_5(G)+\cdots +(k-2)n_k(G)-2$.■

Theorem 2.2

If G is a TSB graph withΔ(G)=k, then $n_2(G) \leq (k-1)n_k+(k-2)n_{k-1}+\cdots +3n_4+2n_3-1$.

Proof. LetG=(V,E) be a TSB graph with Δ(G)=k. In G, if v is a vertex of degree 2 and u,w∈ N(v), then delete the vertex v and join the edge uw.After this vertex deletion and edge joining, degrees of u and w remains the same and number of vertices and edges are decreased by one. We continue this process till all vertices of degree 2 vanish. Let the resultant graph be G'. G' is again a bicyclic graph with $n_i(G')=n_i(G)$ for i ≠ 2 and $n_2(G')=0$. $|E(G')| = \sum_{i=3}^{k} n_i(G') + n_1(G')+1$ = $= \sum_{i=3}^{k} n_i(G) + n_1(G)+1$. Since G is totally segregated $n_2(G) \leq |E(G')|$.
Then $n_2(G) \leq |E(G')| = \sum_{i=3}^{k} n_i(G)+n_1(G)+1$. By theorem 2.1, we have
$n_2(G) \leq (k-1)n_k+(k-2)n_{k-1}+\cdots +3n_4+2n_3-1$.■

Theorem 2.3

If G = (V,E) is a totally segregated ∞- bicyclic graph with Δ(G)=4 and $n_4(G)=1$, then
$n_3(G)-2 \leq n_2(G) \leq 2n_3(G)+2$.

Proof. Let Gbe a totally segregated∞- bicyclic graph. Let w be a common vertex of two cycles in G and $d_G(w)=4$. By theorem 2.2, $n_2(G) \leq 2n_3(G)+2$. In G, if v is a vertex of degree 2 and u,w∈ N(v), then delete the vertex v and join the edge uw.After this vertex deletion and edge joining, degrees of u and w remains the same and number of vertices and edges are decreased by one. We continue this process till all vertices of degree 2 vanish. Let the resultant graph be G'. G' is again a ∞-bicyclic graph with $n_4(G')=1$, $n_3(G')=n_3(G)$, $n_2(G')=0$ and $n_1(G')=n_1(G)$. $|E(G')|=n_3(G')+n_1(G')+2$ and G' has $n_3(G')+n_1(G')+2-(n_1(G')+4)= n_3(G')-2$ balanced edges. Hence $n_2(G) \geq n_3(G')-2$, otherwise G will not be totally segregated graph. But $n_3(G')=n_3(G)$. Hence $n_2(G) \geq n_3(G)-2$. ■

Remark 2.2

In a ∞- bicyclic graph G on n vertices with Δ(G) =4, $n_4(G)=1$.
$n_2 \leq 2n_3+2 \Rightarrow n_3 \geq \frac{n-3}{4}$, $n_2 \leq 2n_3+2 \Rightarrow n_2 \leq \frac{n+1}{2}$
$n_2 \geq n_3-2 \Rightarrow n_3 \leq \frac{n+1}{3}$, $n_2 \geq n_3-2 \Rightarrow n_2 \geq \frac{n-5}{3}$

Theorem 2.4

Let k ≥ 4 be any positive integer, then for any integer x such that k-2 ≤ x ≤ 2k+2, there exists a totally segregated ∞- bicyclic graph with Δ= 4, $n_4=1$, $n_2= x$, $n_3= k$, and $n_1= k$.

Proof. Consider a cycle C_{k-1}on k-1 (k ≥4) vertices and the cycle C_3. Let $w_1 \in V(C_{k-1})$ and
$w_2 \in V(C_3)$. Identify w_1 and w_2 and the identified vertex is named as w. Add pendent edges to all vertices except w. Subdivide all edges of the cycles C_{k-1} and C_3except the edges incident with w. Then subdivide x- (k-2) edges among k+4 edges (k pendent edges and 4 edges incident with w). See Figure 3.The constructed graph is totally segregated ∞- bicyclic graph G when k=5,
$n_2(G)=x=k-2$.■

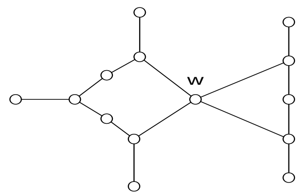

Figure 3

Lemma 2.1

Let G be a totally segregated∞- bicyclic graph on n (n ≥ 9) vertices and Δ(G)=k (k> 3). Then there exists ∞- bicyclic graph, G* on n vertices with Δ(G)=4 and $n_4(G^*)=1$ satisfying the following conditions.

(i)$n_3(G^*) \geq \frac{n_2(G^*)-2}{2}$

(ii)$irr_t(G^*) < irr_t(G)$

Proof.(i) Let G be a totally segregated ∞- bicyclic graph on n(n≥ 9) vertices and Δ (G)=k (k≥4) . Let v_k,v_2 (if exists), $v_1$be any vertex of degree k, 2, 1 respectively in G , and degree sequence of G is given as follows. $(k^{n_k}, (k-1)^{n_{k-1}}, \cdots, 2^{n_2}, 1^{n_1})$. By Theorem 2.2 we have

$n_2(G) \leq (k-1) n_k(G)+(k-2) n_{k-1}(G)+\cdots\cdots+3n_4(G)+2n_3(G)-1$ ……………………(3)

By Theorem 2.1 we have

$n_1(G)=(k-2) n_k(G)+(k-3) n_{k-1}(G)+\cdots\cdots +2n_4(G)+n_3(G)-2$ ………….………….(4)

If $n_2(G) \neq 0$, do the branch-transformation on G from v_k to v_2. If $n_2(G)=0$, do the branch-transformation on G from v_k to v_1. Repeat the branch-transformation on G till maximum degree of resulting graph G* is 4 and $n_4(G^*)=1$.If $N_{Br}(G)$ denotes the number of branch transformations that are done in G ,

$N_{Br}(G)=(k-3) n_k(G)+(k-4) n_{k-1}(G)+\cdots\cdots+2n_5(G)+ n_4(G)-1$ ……………………..……(5)

We prove this lemma by the following two cases:
Case 1.
If $n_2(G) \leq N_{Br}(G)$.That is,

n_2 (G) ≤ (k-3) n_k(G)+(k-4)n_{k-1}(G)+⋯⋯+2n_5(G)+ n_4 (G)-1 ………………..(6)

By equation (6), in the process of branch transformation from v_k to v_2, at some stage, number of vertices of degree two in the resulting bicyclic graph is zero. Then n_2 (G*)=0 or 1.

ie n_2 (G*)≤ 2 n_3 (G*), That is n_3 (G*)≥ $\frac{n_2(G*)}{2}$

Case 2.

If n_2(G) >N_{Br}(G).That is, if

n_2(G)>(k-3)n_k (G)+(k-4)n_{k-1}(G)+⋯⋯+2n_5(G)+ n_4 (G)-1 ……………(7)

When the process of branch transformation ends, equation (3)-(7) gives upper bound for n_2 (G*)

n_2(G*)≤2n_k (G)+2n_{k-1} (G)+⋯⋯+2n_5 (G)+2 n_4 (G)+2n_3(G) ……………..(8)

Number of branch transformations that is done in G and number of vertices of degree greater than or equal to three in G gives n_3(G*).

That is,

n_3 (G*) = (k-3) n_k (G)+(k-4) n_{k-1} (G)+⋯⋯+2n_5 (G)+ n_4 (G)-1

+ n_k (G)+n_{k-1} (G)+⋯⋯+n_5(G)+n_4 (G)+ n_3 (G)

n_3 (G*)=(k-2) n_k(G)+(k-3) n_{k-1}(G)+⋯⋯+3n_5(G)+2 n_4(G)+ n_3(G)-1………….(9)

From equations (8) and (9), n_2 (G*)≤ 2 n_3 (G*)+2

Hence, in both cases, n_3 (G*)≥ $\frac{n_2(G*)-2}{2}$.

(ii) We prove this result by the following two cases.

Case 1.

G" is formed from G by branch transformation on G from v_k to v_2. By this transformation only the number of vertices with degrees k, k-1, 3 and 2 are changed.

Degree sequence ofG is $(k^{n_k}, (k-1)^{n_{k-1}},\cdots\cdots,2^{n_2},1^{n_1})$ and degree sequence of G" is $(k^{n_k-1}, (k-1)^{n_{k-1}+1},\cdots\cdots 3^{n_3+1},2^{n_2-1},1^{n_1})$.

By equation (2)irr_t(G")=irr_t (G)-2n_{k-1} -2n_{k-2} -2n_{k-3} -⋯⋯ -2n_5 -2n_4 -2n_3-2

Hence irr_t (G")<irr_t(G)

Case 2.

G' is formed from G by branch transformation on G from v_k to v_1. By this transformation only the number of vertices with degrees k, k-1, 2 and 1 are changed.

Degree sequence of G is $(k^{n_k}, (k-1)^{n_{k-1}},\cdots\cdots, 3^{n_3},2^{n_2},1^{n_1})$

Degree sequence of G'is $(k^{n_k-1}, (k-1)^{n_{k-1}+1},\cdots\cdots, 3^{n_3},2^{n_2+1},1^{n_1-1})$.

By equation (2)irr_t(G')=irr_t (G)-2n_{k-1} -2n_{k-2} -2n_{k-3} -⋯⋯ -2n_5 -2n_4 -2n_3 -2n_2-2

Hence irr_t (G')<irr_t (G)

We keep on repeating this process and obtain a bicyclic graph G* with Δ=4 and n_4 (G*)=1.

Then irr_t (G*) ≤ irr_t (G) ■

Lemma 2.2

If G* is any ∞- bicyclic graph on n vertices with Δ(G*)=4, n_4(G*)=1 and n_3(G*)≥ $\frac{n_2(G*)-2}{2}$, there exists TSB graph G_1, on n vertices with Δ(G_1) = 4, n_4(G_1) = 1, n_3(G_1)=$\left\lceil\frac{n-3}{4}\right\rceil$and irr_t(G_1≤ irr_t (G*).

Proof. Let G* be a∞- bicyclic graph on n vertices withΔ(G*) = 4, n_4(G*)=1 and n_3(G*)≥ $\frac{n_2(G*)-2}{2}$. ie n_3(G*) ≥ $\frac{n-3}{4}$ by remark 2.2.i.e. n_3 (G*) ≥ $\left\lceil\frac{n-3}{4}\right\rceil$. Let degree sequence of G* be $(4^1, 3^{n_3}, 2^{n_2},1^{n_1})$ and v_3 a vertex of degree three in G*. Let G' be the bicyclic graph obtained from G* by branch transformation on G* from v_3 to v_1 . Degree sequence of G' is $(4^1, 3^{n_3-1}, 2^{n_2+2},1^{n_1-1})$.

Irr_t(G') = (n_3 (G*)-1)+2 (n_2 (G*)+2)+3(n_1 (G*)-1)+(n_3(G*)-1)(n_2 (G*)+2)+
2(n_3 (G*)-1)(n_1 (G*)-1)+ (n_2 (G*)+2) (n_1 (G*)-1)

Irr_t(G') = irr_t (G*)- 2 n_2 (G*)-4

Hence irr_t(G')<irr_t (G*). Repeat the above branch transformation on G* till number of vertices of degree 3 in the resulting graph $G_1$is $\left\lceil\frac{n-3}{4}\right\rceil$. i.e.$n_3$($G_1$)=$\left\lceil\frac{n-3}{4}\right\rceil$, n_1 (G_1)= $\left\lceil\frac{n-3}{4}\right\rceil$,

n_2 (G_1)=n– 1-2$\left\lceil\frac{n-3}{4}\right\rceil$. Here $\left\lceil\frac{n-3}{4}\right\rceil$ -2 ≤ n – 1-2$\left\lceil\frac{n-3}{4}\right\rceil$ ≤ 2$\left\lceil\frac{n-3}{4}\right\rceil$ +2 . By theorem 2.4, for any integer x s.t. k-2 ≤ x ≤ 2k+2, there exists totally segregated ∞- bicyclic graph with Δ=4, n_4 =1, n_2 =x, n_3 =k, n_1 =k. ie there exists totally segregated ∞- bicyclic graph, G_1* on n vertices such that n_3 (G_1*)=$\left\lceil\frac{n-3}{4}\right\rceil$, n_2 (G_1*)=n – 1-2$\left\lceil\frac{n-3}{4}\right\rceil$, n_1 (G_1*)=$\left\lceil\frac{n-3}{4}\right\rceil$, and irr_t (G_1*) ≤ irr_t (G*).■

Theorem 2.5

Let G be a totally segregated ∞- bicyclic graph on n (n ≥ 9) vertices. Then irr_t(G) ≥2n$\left\lceil\frac{n-3}{4}\right\rceil$ -2$\left\lceil\frac{n-3}{4}\right\rceil^2-2\left\lceil\frac{n-3}{4}\right\rceil$+2n-2, and the equality holds if and only if Δ(G)=4, n_4(G)=1, n_3(G)= $\left\lceil\frac{n-3}{4}\right\rceil$.

Corollary 2.1

Let G be a totally segregated ∞- bicyclic graph on n vertices. Then,

1. irr_t (G) ≥ $\frac{3n^2}{8}$+$\frac{3n}{2}$-2, when n=4k, k=3,4,⋯ . Equality holds if and only if n_4(G)=1, n_3(G)=k, n_2 (G)=2k-1, n_1(G)=k .
2. irr_t(G) ≥ $\frac{3n^2}{8}$+$\frac{5n}{4}$-$\frac{13}{8}$, when n=1+4k, k=2,3,4,⋯ . Equality holds if and only if n_4(G)=1, n_3(G)=k,n_2(G)=2k, n_1(G)=k.
3. irr_t (G)≥ $\frac{3n^2}{8}$+n-$\frac{3}{2}$, When n=2+4k, k=2,3,4,⋯ . Equality holds if and only if n_4(G)=1, n_3(G)=k, n_2 (G)=2k+1, n_1 (G)=k.
4. irr_{t} (G)≥ $\frac{3n^2}{8}$+$\frac{3n}{4}$-$\frac{13}{8}$, when n=3+4k, k=2,3,4,\cdots . Equality holds if and only if n_4 (G)=1, n_3(G)=k,n_2(G)=2k+2, n_1 (G)=k ■

Total irregularity is completely determined by its degree sequence. Degree sequence of totally segregated∞- bicyclic graph with minimum Total Irregularity (min G_n) for each n is given below:

No. of vertices: n	Degree sequence of min G_n
n = 4k, k = 2,3,⋯	$(4^1,3^k,2^{2k-1},1^k)$
n = 4k+1, k = 2,3,⋯	$(4^1,3^k,2^{2k},1^k)$
n = 4k+2, k = 2,3,⋯	$(4^1,3^k,2^{2k+1},1^k)$
n = 4k+3, k = 2,3,⋯	$(4^1,3^k,2^{2k+2},1^k)$

Remark 2.3

For n ≤ 6 totally segregated ∞- bicyclic graph onn vertices does not exist.

For n=7,8totally segregated ∞- bicyclic graph on n vertices with minimum total irregularity is presented in Figure 4.

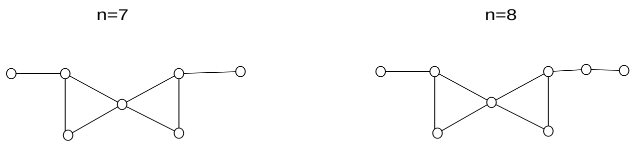

Figure 4

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