Fixed Point Theorems of Soft Metric Space Using Altering Distance Function

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Abstract: In the present paper, some fixed point theorems are proved through rational expression in altering distance functions and property P for the contraction mappings.

Keywords: - Soft metric space, Altering distance function, fixed point.

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I. INTRODUCTION & PRELIMINARIES

The fixed point theorems in metric spaces are playing a major role to solve many problems in a mathematical analysis. So the attraction of metric spaces to a large numbers of mathematicians is understandable.

Altering distance function for self-mapping on a metric space established by M. S. Khan in 1984 and it can be expanded by M. Swalesh and S. Seesa [7] that they introduced a control function which they called as altering distance function in the research of fixed point theory.

In the year 1999, Molodtsov [11] initiated a novel concept of soft sets theory as a new mathematical tool for dealing with uncertainties. A soft set is a collection of approximate descriptions of an object. Soft systems provide a very general framework with the involvement of parameters. Since soft set theory has a rich potential, applications of soft set theory in other disciplines and real life problems are progressing rapidly.

Maji et al. [8, 9] worked on soft set theory and presented an application of soft sets in decision making problems. Chen [2] introduced a new definition of soft set parameterization reduction and a comparison of it with attribute reduction in rough set theory. Many researchers contributed towards many structure on soft set theory. [1,2]. M. Shabir and M. Naz [12] presented soft topological spaces and they investigated some properties of soft topological spaces. Later, many researches about soft topological spaces were studied in [6,10,12,13]. In these studies, the concept of soft point is expressed by different approaches. In the study we use the concept of soft point which was given in [4, 13].

Definition 2.1: Let $X$ be an initial universe set and $E$ be a set of parameters. A pair $(F, E)$ is called a soft set over $X$ if and only if $X$ is a mapping from $E$ into the set of all subsets of the set $X$, i.e., $F: E \rightarrow P(X)$, where $P(X)$ is the power set of $X$.

Definition 2.2: The intersection of two soft sets $(F,A)$ and $(G,B)$ over $X$ is the soft set $(H,C)$, where $C = A \cap B$ and $\forall \varepsilon \in C, H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$. This is denoted by $(F, A) \cap (G, B) = (H, C)$.

Definition 2.3: The union of two soft sets $(F,A)$ and $(G,B)$ over $X$ is the soft set, where $C = A \cup B$ and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if } \varepsilon \in A - B \\ G(\varepsilon), & \text{if } \varepsilon \in B - A \\ F(\varepsilon) \cup G(\varepsilon), & \varepsilon \in A \cap B \end{cases}$$

This relationship is denoted by $(F, A) \cup (G, B) = (H, C)$.

Definition 2.4: The soft set $(F,A)$ over $X$ is said to be a null soft set denoted by $\Phi$ if for all $\varepsilon \in A, F(\varepsilon) = \phi$ (null set).

Definition 2.5: A soft set $(F,A)$ over $X$ is said to be an absolute soft set, if for all $\varepsilon \in A, F(\varepsilon) = X$.

Definition 2.6: The difference of two soft sets $(F,A)$ and $(G,B)$ over $X$ denoted by $(H,E)$, is defined as $H(\varepsilon) = F(\varepsilon) \setminus G(\varepsilon)$ for all $\varepsilon \in E$.

Definition 2.7: The complement of a soft set $(F,A)$ is denoted by $(F,A)^{c}$ and is defined by $(F,A)^{c} = (F^{c}, A)$ where $F^{c}: A \rightarrow P(X)$ is mapping given by $F^{c}(a) = X - F(a), \forall a \in A$.

Definition 2.8: Let $\mathbb{R}$ be the set of real numbers and $B(\mathbb{R})$ be the collection of all nonempty bounded subsets of $\mathbb{R}$ and $E$ taken as a set of parameters. Then a mapping $F:E \rightarrow B(\mathbb{R})$ is called a soft real set. It is denoted by $(F,E)$. If specifically $(F,E)$ is a singleton soft set, then identifying $(F,E)$ with the corresponding soft element, it will be called a soft real number and denoted $\tilde{F}, \tilde{\tilde{F}}, \tilde{\varepsilon}$ etc.

$\tilde{0}, \tilde{1}$ are the soft real numbers where $\tilde{0}(\varepsilon) = 0, \tilde{1}(\varepsilon) = 1$ for all $\varepsilon \in E$, respectively.
Definition 2.9: For two soft real numbers

(i) \( r \leq s \) if \( r(e) \leq s(e) \), for all \( e \in E \).

(ii) \( r \geq s \) if \( r(e) \geq s(e) \), for all \( e \in E \).

(iii) \( r < s \) if \( r(e) < s(e) \), for all \( e \in E \).

(iv) \( r > s \) if \( r(e) > s(e) \), for all \( e \in E \).

Definition 2.10: A soft set over \( X \) is said to be a soft point if there is exactly one \( e \in E \) such that \( P(e) = \{x\} \) for some \( x \in X \) and \( P(e) = \phi \), \( \forall e' \in E \setminus \{e\} \). It will be denoted by \( x_e \).

Definition 2.11: Two soft points \( x_e, x_{e'} \) are said to be equal if \( e = e' \) and \( P(e) = P(e') \) i.e. \( x = y \). Thus \( x_e \neq x_{e'} \Leftrightarrow e \neq y \) or \( e \neq e' \).

Definition 2.12: A mapping \( d: SP(X) \times SP(X) \rightarrow \mathbb{R}(E)^* \), is said to be a soft metric on the soft set \( X \) if \( d \) satisfies the following conditions:

(M1) \( d(x_{e_1}, y_{e_2}) \geq 0 \) for all \( x_{e_1}, y_{e_2} \in X \).

(M2) \( d(x_{e_1}, y_{e_2}) = 0 \) if and only if \( x_{e_1} = y_{e_2} \).

(M3) \( d(x_{e_1}, y_{e_2}) = d(y_{e_2}, x_{e_1}) \) for all \( x_{e_1}, y_{e_2} \in X \).

(M4) \( d(x_{e_1}, y_{e_2}) \leq d(x_{e_1}, z_{e_3}) + d(z_{e_3}, y_{e_2}) \) for all \( x_{e_1}, y_{e_2}, z_{e_3} \in X \).

The soft set \( X \) with a soft metric \( d \) on \( X \) is called a soft metric space and denoted by \((X, d, E)\).

Definition 2.13 (Cauchy Sequence): A sequence \( \{x_{e_i}\}_{i \geq 1} \) of soft points in \((X, d, E)\) is considered as a Cauchy sequence in \( X \) if corresponding to every \( \varepsilon > 0 \), \( \exists m \in \mathbb{N} \) such that \( d(x_{e_i}, x_{e_j}) \leq \varepsilon \), \( \forall i, j \geq m \), i.e. \( d(x_{e_i}, x_{e_j}) \rightarrow 0 \), as \( i, j \rightarrow \infty \).

Definition 2.14 (Complete Soft Metric Space): A soft metric space \((X, d, E)\) is called complete, if every Cauchy Sequence in \( X \) converges to some point of \( X \).

Definition 2.15: A function \( \psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is called an altering distance function if the following properties are satisfied:

i. \( \psi(0) = 0 \Leftrightarrow t = 0 \).

ii. \( \psi \) is monotonically non decreasing.

iii. \( \psi \) is continuous.

By \( \psi \) denotes the set of all altering distance function.

Definition 2.16: Let \((X, d, E)\) be a complete soft metric space for a self-mapping \((f, \phi)\) with a nonempty set \( E(f, \phi) \). Then \((f, \phi)\) is said to satisfy the property \( P \) if \( E(f, \phi)^n \) for each \( n \in \mathbb{N} \).

Theorem 2.1: Let \((X, d, E)\) be a complete soft metric space, and mapping \((f, \phi)\) satisfies the following inequality:

\[ \psi \left[ d((f, \phi)(x), (f, \phi)(y)) \right] \leq d((f, \phi)(x), (f, \phi)(y)) \]

Thus \((f, \phi)\) has a unique fixed point in \( X \) and moreover for each \( x \in X \), \( \lim_{n \rightarrow \infty} (f, \phi)^n x = y \).

Theorem 2.2: Let \((X, d, E)\) be a complete soft metric space, and mapping \((f, \phi)\) satisfies the following inequality:

\[ \psi \left[ d((f, \phi)(x), (f, \phi)(y)) \right] \leq \alpha \psi \left[ d(x, y) \right] \]

Thus \((f, \phi)\) has a unique fixed point in \( X \) and moreover for each \( x \in X \), \( \lim_{n \rightarrow \infty} (f, \phi)^n x = y \).

II. MAIN RESULTS

Theorem 3.1: Let \((X, d, E)\) be a complete soft metric space, and mapping \((f, \phi)\) satisfies the following inequality:

\[ \psi \left[ d((f, \phi)(x), (f, \phi)(y)) \right] \leq \alpha \psi \left[ d(x, y) \right] \]

Thus \((f, \phi)\) has a unique fixed point in \( X \) and moreover for each \( x \in X \), \( \lim_{n \rightarrow \infty} (f, \phi)^n x = y \).

Proof: Let \( x_0 \) be any soft point in \( SP(X) \).
Set \( x_{\lambda_1}^1 = (f, \varphi)(x_0^0) = (f(\tilde{x}_0^0))_{\varphi(\lambda)} \)

Since \( \frac{(a+b)}{(1-b)} < 1 \) from (3.1.2), we have

\[
\lim_{n \to \infty} \psi[\tilde{d}(\tilde{x}_{\lambda_n}^n, \tilde{x}_{\lambda_{n+1}}^{n+1})] = 0
\]

From the result given that \( \psi \in \Psi \), we have

\[
\lim_{n \to \infty} \tilde{d}(\tilde{x}_{\lambda_n}^n, \tilde{x}_{\lambda_{n+1}}^{n+1}) = 0
\]

… (3.1.3)

Now we will show that \( \{\tilde{x}_{\lambda_n}^n\} \) is Cauchy sequence in \( \tilde{X} \). Suppose that \( \{\tilde{x}_{\lambda_n}^n\} \) is not a Cauchy sequence, which means that there is a constant \( \epsilon_0 > 0 \) such that for each positive integer \( k \), there exist a positive integer \( n \) and \( \lambda_n \) with 

\[
\exists \epsilon_0 \forall k \exists n \exists \lambda_n \text{ such that } \tilde{d}(\tilde{x}_{\lambda_n}^n, \tilde{x}_{\lambda_{n+1}}^{n+1}) > \epsilon_0
\]

By triangle inequality

\[
\epsilon_0 \leq \tilde{d}(\tilde{x}_{\lambda_m(k)}^m, \tilde{x}_{\lambda_{m(k)-1}}^{m(k)-1}) \leq \tilde{d}(\tilde{x}_{\lambda_m(k)}^m, \tilde{x}_{\lambda_m(k)-1}^{m(k)-1}) + \tilde{d}(\tilde{x}_{\lambda_m(k)-1}^{m(k)-1}, \tilde{x}_{\lambda_m(k)-1}^{m(k)-1})
\]

Letting \( k \to \infty \) and using (3.1.3), we have

\[
\lim_{k \to \infty} \tilde{d}(\tilde{x}_{\lambda_m(k)}^m, \tilde{x}_{\lambda_{m(k)}-1}^{m(k)-1}) = \epsilon_0
\]

… (3.1.4)

Similarly, we have

\[
\lim_{n \to \infty} \tilde{d}(\tilde{x}_{\lambda_{m(k)+1}}^{m(k)+1}, \tilde{x}_{\lambda_{m(k)+1}}^{m(k)+1}) = \epsilon_0
\]

… (3.1.5)

Putting \( \tilde{x}_\lambda = \tilde{x}_{\lambda_m(k)}^m \) and \( \tilde{y}_\mu = \tilde{x}_{\lambda_n}^n \) in (3.1.1) we have

\[
\psi \left[ \frac{(a+b)}{(1-b)} \right] \psi \left[ \frac{(a+b)}{(1-b)} \right] \psi \left[ \frac{(a+b)}{(1-b)} \right]
\]
Fixed Point Theorems of Soft Metric Space Using Altering Distance Function

\[ \psi \left[ d \left( \tilde{x}_{m(k)+1}, \tilde{x}_{n(k)+1} \right) \right] = \psi \left[ d \left( f, \phi \right) \left( \tilde{x}_{m(k)}, \tilde{x}_{n(k)} \right) \right] \leq \tilde{\alpha} \psi \left[ d \left( \tilde{x}_{m(k)}, \tilde{x}_{n(k)} \right) \right] \]

Using (3.1.3), (3.1.4) and (3.1.5) we have

\[ \psi [\varepsilon_0] = \lim_{n \to \infty} \psi \left[ d \left( \tilde{x}_{m(k)+1}, \tilde{x}_{n(k)+1} \right) \right] \leq \tilde{\alpha} \psi [\varepsilon_0] \]

Since \( \tilde{\alpha} \in (0,1) \), we get a contradiction. Thus \( \{ \tilde{x}_{m(k)} \} \) is a Cauchy sequence in the complete soft metric space \( \tilde{X} \). Thus there exist \( \tilde{x}_n \in \tilde{X} \) such that \( \tilde{x}_{m(k)} \to \tilde{x}_n \), \( n \to \infty \).

Putting \( \tilde{x}_n = \tilde{x}_{m(k)} \) and \( \tilde{y}_n = \tilde{x}_n \) in (3.1.1) we have

\[ \psi \left[ d \left( \tilde{x}_{m(k)+1}, \tilde{x}_{n(k)+1} \right) \right] = \psi \left[ d \left( f, \phi \right) \left( \tilde{x}_{m(k)}, \tilde{x}_{n(k)} \right) \right] \leq \tilde{\alpha} \psi \left[ d \left( \tilde{x}_{m(k)}, \tilde{x}_{n(k)} \right) \right] \]

Therefore

\[ \psi \left[ d \left( \tilde{x}_n, \tilde{y}_n \right) \right] \leq \tilde{\alpha} \psi \left[ d \left( \tilde{x}_n, \tilde{y}_n \right) \right] \]

Since \( \tilde{\alpha} \in (0,1) \), then \( \psi \left[ d \left( \tilde{x}_n, \tilde{y}_n \right) \right] = 0 \), which implies that \( d \left( \tilde{x}_n, \tilde{y}_n \right) = 0 \).

Thus \( \tilde{x}_n = \tilde{y}_n \).

Now we are going to establish the uniqueness of the fixed point. Let \( \tilde{x}_1, \tilde{y}_1 \) be two fixed point of \( (f, \phi) \) such that \( \tilde{x}_1 \neq \tilde{y}_1 \).

Putting \( \tilde{x} = \tilde{x}_1 \) and \( \tilde{y} = \tilde{y}_1 \) in (3.1.1) we have

\[ \psi \left[ d \left( \tilde{x}_n, \tilde{y}_n \right) \right] = \psi \left[ d \left( \left( f, \phi \right)(\tilde{x}_n), \left( f, \phi \right)(\tilde{y}_n) \right) \right] \leq \tilde{\alpha} \psi \left[ d \left( \tilde{x}_n, \tilde{y}_n \right) \right] \]

Thus \( \tilde{x}_n = \tilde{y}_n \).

Corollary 3.2: Let a complete soft metric space \( (\tilde{X}, d, E) \) and let \( (f, \phi) : (\tilde{X}, d, E) \to (\tilde{X}, d, E) \) be a mapping. We assume that for each \( \tilde{x} \), \( \tilde{y} \in \tilde{X} \),

\[ \int_0^\infty \varphi(t) dt \leq \tilde{\alpha} \int_0^\infty \varphi(\tilde{x} \cdot \mu(t)) \varphi(t) dt \]

where \( 0 < \tilde{\alpha} + 2\tilde{b} < 1 \) and \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue integrable mapping which is Summable on each compact subset of \( [0, \infty) \), non negative and such that

\[ \int_0^\infty \varphi(t) dt > 0, \text{ for all } \epsilon > 0. \]

Then \( (f, \phi) \) has a unique fixed point in \( \tilde{x} \in \tilde{X} \) such that \( \lim_{n \to \infty} (f, \phi)^n \tilde{x} = \tilde{x} \), for each \( \tilde{x} \in \tilde{X} \).

Proof: Let \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a mapping as we define \( \psi(t) = \int_0^t \psi(t') dt', t \in \mathbb{R}^+ \). It is clear that \( \psi(t) = 0 \) and \( \psi \) is monotonically non decreasing and by hypothesis \( \Psi \) is absolutely continuous. Hence \( \psi \) is continuous. Therefore, \( \psi \in \Psi \), so by (3.1.1) becomes

\[ \psi \left[ d \left( \left( f, \phi \right)(\tilde{x}_n), \left( f, \phi \right)(\tilde{y}_n) \right) \right] \leq \tilde{\alpha} \psi \left[ d \left( \tilde{x}_n, \tilde{y}_n \right) \right] \]

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Hence from Theorem 3.1 there exist a unique fixed point \( \hat{y}_\lambda^* \in \hat{X} \) such that for each \( \tilde{y}_\mu \in \hat{X}, \lim_{n \to \infty} (f, \varphi)^n \tilde{y}_\mu = \hat{y}_\lambda^* \).

In this section we are going to prove that the mappings satisfying the contractive condition [2.1.1],[2.2.1] and [3.1.1] fulfil the property \( P \).

**Theorem 3.3:** Let \((\hat{X}, d, E)\) be a complete soft metric space, we have \( \psi \in \Psi \). Let mapping \((f, \varphi): (\hat{X}, d, E) \to (\hat{X}, d, E)\) satisfies the following condition:

\[
\psi \left[ d \left( (f, \varphi)(\tilde{x}_\lambda), (f, \varphi)(\tilde{y}_\mu) \right) \right] \leq \tilde{a} \psi [d(\tilde{x}_\lambda, \tilde{y}_\mu)]
\]

For all \( \tilde{x}_\lambda, \tilde{y}_\mu \in \tilde{X} \), and for some \( 0 < \tilde{a} < 1 \). Then \( E_{(f, \varphi)} \neq \emptyset \) and \((f, \varphi)\) has a property \( P \).

**Proof:** From Theorem [2.1], \((f, \varphi)\) has a fixed point therefore \( E_{(f, \varphi)} \neq \emptyset \) for every \( n \in N, n > 1 \) and we assume that \( \hat{y}_\lambda^* \in E_{(f, \varphi)} \) we have to prove that \( \hat{y}_\lambda^* \in E_{(f, \varphi)} \).

Assume that \( \hat{y}_\lambda^* \neq (f, \varphi)(\hat{y}_\lambda^*) \), from [2.1.1] we have

\[
\psi(d((f, \varphi)(\hat{y}_\lambda^*), (f, \varphi)(\hat{y}_\lambda^*)))) = \psi(d((f, \varphi)^{n}(\hat{y}_\lambda^*), (f, \varphi)^{n+1}(\hat{y}_\lambda^*))))
\]

\[
\leq \tilde{a} \psi(d((f, \varphi)^{n-1}(\hat{y}_\lambda^*), (f, \varphi)^{n}(\hat{y}_\lambda^*)))) \leq \cdots \leq \tilde{a}^n \psi(d((f, \varphi)(\hat{y}_\lambda^*), (f, \varphi)(\hat{y}_\lambda^*))))
\]

Since \( \tilde{a} \in (0,1) \), \( \lim_{n \to \infty} \psi(d((f, \varphi)(\hat{y}_\lambda^*), (f, \varphi)(\hat{y}_\lambda^*)))) = 0 \).

From the face that, \( \psi \in \Psi \) we get \( \hat{y}_\lambda^* = (f, \varphi)(\hat{y}_\lambda^*) \) which is a contradiction.

Therefore \( \hat{y}_\lambda^* \in E_{(f, \varphi)} \) i.e. \((f, \varphi)\) has a property \( P \).

**Theorem 3.4:** Let \((\hat{X}, d, E)\) be a complete soft metric space, and let mapping \((f, \varphi): \hat{X}, d, E \to \hat{X}, d, E\) satisfies the contractive condition:

\[
\tilde{d} \left( (f, \varphi)(\tilde{x}_\lambda), (f, \varphi)(\tilde{y}_\mu) \right) \leq \tilde{a} \left[ \tilde{d}(\tilde{x}_\lambda, \tilde{y}_\mu) \right] + \tilde{b} \left[ \tilde{d}(\tilde{x}_\lambda, \tilde{y}_\mu) \right] + \tilde{c} \left[ \tilde{d}(\tilde{x}_\lambda, \tilde{y}_\mu) \right] + \tilde{d} \left[ \tilde{d}(\tilde{x}_\lambda, \tilde{y}_\mu) \right]
\]

For all \( \tilde{x}_\lambda, \tilde{y}_\mu \in \tilde{X} \), and \( \tilde{a} > 0, \tilde{b} > 0, \tilde{a} + 2\tilde{b} < 1 \). Then \( E_{(f, \varphi)} \neq \emptyset \) and \((f, \varphi)\) has a property \( P \).

**Proof:** From Theorem [2.2], \( E_{(f, \varphi)} \neq \emptyset \) therefore \( E_{(f, \varphi)} \neq \emptyset \) for every \( n \in N, n > 1 \) and we assume that \( \hat{y}_\lambda^* \in E_{(f, \varphi)} \) we have to prove that \( \hat{y}_\lambda^* \in E_{(f, \varphi)} \).

Assume that \( \hat{y}_\lambda^* \neq (f, \varphi)(\hat{y}_\lambda^*) \), from [2.2.1] we have

\[
\tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*)) = \tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*)) + \tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*)) + \tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*)) + \tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*)) + \tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*))
\]

\[
\leq \tilde{a} \tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*)) + \tilde{b} \tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*)) + \tilde{c} \tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*)) + \tilde{d} \tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*))
\]

\[
\leq \cdots \cdots \leq \left( \frac{\tilde{a}}{1-\tilde{b}} \right)^n \tilde{d}(\tilde{y}_\lambda^*, (f, \varphi)(\tilde{y}_\lambda^*))
\]

Which is the contradiction. Consequently \( \hat{y}_\lambda^* \in E_{(f, \varphi)} \) and \((f, \varphi)\) has a property \( P \).

**Theorem 3.5:** Let \((\hat{X}, d, E)\) be a complete soft metric space and let \( \psi \in \Psi \). Let \((f, \varphi): \hat{X}, d, E \to \hat{X}, d, E\) be a mapping satisfies the contractive condition:
\[\psi\left[\overline{d}\left((f, \varphi)(x_{\lambda}), (f, \varphi)(y_{\mu})\right)\right] \leq \tilde{a}\psi\left[\overline{d}(x_{\lambda}, y_{\mu})\right] \leq \tilde{a}\psi\left[\overline{d}(x_{\lambda}, y_{\mu})\right] \leq \frac{\alpha + b\psi}{\alpha + b}\] 

For all \(x_{\lambda}, y_{\mu} \in \bar{X}\), and \(\tilde{a} > 0, \tilde{b} > 0, \tilde{a} + 2\tilde{b} < 1\). Then \(E_{(f, \varphi)} \neq \phi\) and \((f, \varphi)\) has a property \(P\).

**Proof:** From Theorem [2.1], \((f, \varphi)\) has a fixed point.

Therefore \(E_{(f, \varphi)} = \phi\) for every \(n \in N, n > 1\) and

We assume that \(y_{\lambda} \in E_{(f, \varphi)}\) we have to prove that \(y_{\lambda} \in E_{(f, \varphi)}\)

Assume that \(y_{\lambda} \neq (f, \varphi)y_{\lambda}\).

\[\psi[\overline{d}(y_{\lambda}, (f, \varphi)y_{\lambda})] = \psi[\overline{d}((f, \varphi)^n(y_{\lambda}), (f, \varphi)^{n+1}(y_{\lambda}))] \leq \tilde{a}\psi[\overline{d}((f, \varphi)^{n-1}(y_{\lambda}), (f, \varphi)^n(y_{\lambda}))] \]

\[\leq \tilde{a}\psi[\overline{d}((f, \varphi)^{n-1}(y_{\lambda}), (f, \varphi)^n(y_{\lambda}))] \leq \tilde{a}\psi[\overline{d}((f, \varphi)^{n-1}(y_{\lambda}), (f, \varphi)^n(y_{\lambda}))] \leq \frac{\alpha + b\psi}{\alpha + b}\]

This is a contradiction. Therefore \(\psi[\overline{d}(y_{\lambda}, (f, \varphi)y_{\lambda})] = 0\), since \(\psi \in \Psi\)

This conclude that \(\overline{d}(y_{\lambda}, (f, \varphi)y_{\lambda}) = 0\), thus \(y_{\lambda} \in E_{(f, \varphi)}\) and \((f, \varphi)\) has the property \(P\).

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