Hybrid Quasi-Newton with New Conjugate Gradient using Exact Line Search

Ummie Khalthum Mohd Yusof, Mohd Asrul Hery Ibrahim, Mohd Rivaie, Mustafa Mamat, Mohamad Afendee Mohamed, Puspa Liza Ghazali

Abstract: Until now, Quasi-newton (QN) method is the most well-known method for solving unconstrained optimization problem. This method consumes lesser time as compared to Newton method since it is unnecessary to compute Hessian matrices. For QN method, BFGS is the best solver in finding the optimum solutions. Therefore, a new hybrid coefficient which possesses the convergence analysis computed by exact line search is introduced. This new hybrid coefficient is numerically proven by producing the best outcomes with least iteration number and CPU time.

Index Terms: Quasi-Newton method, sufficient descent condition and global convergence, unconstrained optimization.

I. INTRODUCTION

QN methods was developed based on Newton’s method, and it is still used for solving unconstrained optimization problems. The most popular quasi-Newton method is BFGS. In 1970, BFGS was suggested independently by Broyden, Fletcher, Goldfarb and Shanon after Rank One in 1959 and DFP formula in 1963. Rank one and DFP formula have the tendency in solving the larger non-quadratic problems. The BFGS method was an improvement of both rank one and DFP formula [8]-[11], [13]. This formula is known to robust with the sloppy line search. Next, the conjugate gradient algorithm is which known as an iterative method may produces better approximation to the minimum of a general unconstrained nonlinear function of $N$ variables at each iteration. This algorithm initiated by Hestenes-Stiefel in 1952. With the simple algorithm, the conjugate gradient method is easy to be implemented. The main target of unconstrained optimization is quoted as

$$
\min_{m \in \mathbb{R}^n} f(m)
$$

that is finding the minimum value for a function $f(m)$ such that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable and $\mathbb{R}^n$ represents n-dimensional Euclidean space and. The iterative methods are used to solve (1). For each $q^{th}$ iteration, an approximation point, $m_q$ and the $(q+1)^{th}$ iteration is related as

$$
m_{q+1} = m_q + \alpha_q d_q
$$

where $m_{q+1}, \alpha_q, d_q$ each represent the new iterate point, step size and search direction respectively. The $d_q$ considered is as

$$
d_q = \begin{cases} 
-H_q g_q & \text{if } m = 0 \\
-H_q g_q + \eta (-g_q + \beta_q d_{q-1}) & \text{if } m \geq 1
\end{cases}
$$

where $g_q$ is a gradient of $f(m)$ at point $m_q$ and $\beta_q$ is CG coefficient.

Previous studies gives valuable intentions to latest researchers come out with new CG coefficient that gain ideas from the classical $\beta_q$ as listed in Table 1.

Table 1: CG coefficients

<table>
<thead>
<tr>
<th>CG Coefficients</th>
<th>Years</th>
</tr>
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<tbody>
<tr>
<td>$\beta_q^{HS}$</td>
<td>$g_q g_q - q - q_{q-1}$</td>
</tr>
<tr>
<td>$\beta_q^{FR}$</td>
<td>$g_q g_q$</td>
</tr>
<tr>
<td>$\beta_q^{PRP}$</td>
<td>$g_q g_q - q - q_{q-1}$</td>
</tr>
</tbody>
</table>

where $g_q$ and $g_{q-1}$ are gradients of $f(m)$ at points $m_q$ and $m_{q-1}$ respectively, while $d_{q-1}$ is search direction for the previous iteration and $\|\cdot\|$ is a norm of vectors.

In recent times, researchers keep developing new ideas to overcome the disadvantages of previous formula as in [2], [5], [6], [12], [15], [17]-[19], [20], [23].

The search direction for QN method is the solution of linear system

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where $H_q$ is known as approximation of Hessian, while initial matrix $H_0$ is determined by the identity matrix, and successively updated by some formula. This research uses a BFGS formula in a classical algorithm. The update formula for BFGS is

$$H_{q+1} = H_q - \frac{H_q s_q y_q T}{s_q^T H_q s_q} + \frac{y_q y_q T}{s_q y_q}$$

with $s_q = m_q - m_{q-1}$ and $y_q = s_q - s_{q-1}$. The approximation that the Hessian must fulfill is the curvature condition as below

$$H_{q+1} s_q = y_q$$

Instead of using the inexact line search, this paper focuses on BFGS algorithm with exact line search such that

$$\min_{d_q \geq 0} f(m_q + \alpha q d_q)$$

### II. THE NEW COEFFICIENT

#### A. Propose New Coefficient

In solving unconstrained optimization problems, the calculation of the search direction determines the method to be used. The modification of the search direction has already been introduced by previous researchers [21]. By focusing on QN and CG methods, another new hybrid method is introduced by applying the $\beta^{UAM}$ (Ummie, Asrul and Mustafa) and denoted as BFGS-UAM [8], [9]. The idea on proposing this new $\beta_q$ is based on previous researchers [7], [22]. The new CG coefficient is

$$\beta^{UAM}_{q+1} = \max \left\{0, \frac{T_{q+1} \left\{ \begin{array}{c} T \left( g_{q+1} - \|g_q\| \right) g_q \\ \|g_q\|^2 \end{array} \right\}} \right\}$$

Hence, the complete algorithms for the HBFGS methods will be arranged below:

Step 0: Given a starting point $m_0$ and $H_0$, select the values for $s$, $\beta$ and $\alpha$ and then fix $q = 1$.
Step 1: Exit if $\|g(m_{q+1})\| < 10^{-6}$ or $m \geq 1000$.
Step 2: Calculate the value of search direction by (3)
Step 3: Compute the value of step size $\alpha_q$ by (7)

Step 4: Compute the difference between $s_q = m_q - m_{q-1}$ and $y_q = s_q - s_{q-1}$.
Step 5: Update $H_q s_q$ by (5) to obtain $H_q$
Step 6: Set $m = m + 1$ and return to Step 1.

### III. CONVERGENCE ANALYSIS

The convergent properties of this method clearly presented in this section with assumption and theorem. Sufficient descent condition and global convergence are established to ensure an algorithm is verified for both conditions.

#### A. Sufficient Descent Condition

For all $q \geq 0$, every search direction $d_q$ satisfied the descent property

$$s_q d_q < 0$$

The search direction is said to satisfy the sufficient descent condition if there exists a constant $c_1 > 0$ such that

$$s_q d_q \leq c_1 \| g_q \|^2$$

for all $q \geq 0$.

**Theorem 1.** Suppose a hybrid method having search direction (3) and $\beta^{UAM}$ given as (8), then condition (11) will holds for all $q \geq 0$.

**Proof.** Based on (12), we observe that

$$s_q d_q = \frac{s_q T}{H_q g_q + \eta(-g_q + \beta_q d_{q-1})}$$

$$= -s_q H_q g_q - \eta g_q^3 + \eta \beta_q s_q d_{q-1}$$

since the line is exact, then $s_q d_{q-1} = 0$. $H_q = \lambda$ and thus,

$$s_q d_q \leq \lambda \| g_q \|^2 - \eta \| g_q \|^2$$

$$\leq (\lambda - \eta) \| g_q \|^2$$

$$\leq (\lambda + \eta) \| s_q \|^2$$

$$\leq c_1 \| s_q \|^2$$

where $c_1 = \lambda$ which is bound away from zero. Therefore, $s_q d_q \leq c_1 \| g_q \|^2$ holds and thus complete the proof.
B. Global Convergence Properties

Importantly, to make sure that there are remain not less than zero to examine the global convergence, then able simplify our new \( \rho_{q}^{\text{UAM}} \) as follows:

\[
\rho_{q+1}^{\text{UAM}} = \begin{cases} 
0 & \text{for } \|g_{q+1}\|^2 < \|g_{q}\|^T g_{q+1}g_{q} \\
\|g_{q+1}\|^2 & \text{for } \|g_{q+1}\|^2 \geq \|g_{q}\|^T g_{q+1}g_{q} 
\end{cases}
\]

(13)

Hence,

\[
0 \leq \rho_{q+1}^{\text{UAM}} \leq \frac{\|g_{q+1}\|^2}{\|g_{q}\|^2} \quad \text{for all } q.
\]

(14)

A few assumptions are needed to follow the objective function.

**Assumption 1.** Consider the following:

i. The objective function \( f(x) \) is twice continuously differentiable.

ii. The level set L satisfies convex property. Let two positive constants \( c_1 \) and \( c_2 \) satisfying \( c_1 \|z\|^2 \leq z^T F(m)z \leq c_2 \|z\|^2 \) for all \( z \in \mathbb{R}^n \) where \( F(m) \) is the Hessian matrix for \( f \).

iii. The Hessian matrix is Lipschitz continuous at the point \( m^* \) such that there exists the positive constant \( c_3 \) satisfying all \( m \) in a neighborhood of \( m^* \).

Under this assumption, the following Lemma are needed, earlier shown true by Zoutendijk.

**Lemma 1.** Let the Assumption 1 be true. By considering an iterative CG method and search direction of hybrid method, with \( d_q \) as a search direction and \( \alpha_q \) satisfies the exact line search. Then, the following condition known as the Zoutendijk condition holds

\[
\sum_{q=0}^{\infty} \left( \frac{g_q^T d_q}{\|d_q\|^2} \right)^2 < \infty
\]

**Theorem 1.** [24]

Consider a sequence \( \{B_q\} \) generated using BFGS formula, where \( B_q \) is symmetric and positive definite, and \( y_q^T s_q > 0 \) for all \( q \). In addition, consider \( \{s_q\} \) and \( \{y_q\} \) such that

\[
\left\| \frac{y_q - Gs_q}{s_q} \right\| \leq \epsilon
\]

For some symmetric and positive definite matrix \( G(m^*) \) and for some sequence \( \{\epsilon_q\} \) with the property \( \sum_{q=1}^{\infty} \epsilon_q < \infty \).

Then,

\[
\lim_{q \to \infty} \frac{\|B_q - G_q\|}{\|s_q\|^2} = 0 \quad \text{and the sequence } \left\{\frac{B_q}{\|s_q\|^2}\right\} \text{ are bound.}
\]

**Theorem 2 (global convergence)**

Let Assumption 1 and Theorem 1 be true. Consider any \( f_q^{\text{UAM}} \) method in the form of (3) and (2) where \( \alpha_q \) is a result of an exact line search (7) and the sufficient descent condition also holds. Then, (15) is true.

\[
\lim_{q \to \infty} \|g_q\|^2 = 0 \quad \text{or} \quad \sum_{q=0}^{\infty} \|s_q\|^4 < \infty \quad \text{(15)}
\]

**Proof:** Proof by contradiction. In other words, if Theorem 2 does not hold, then the exists a constant \( \epsilon > 0 \) such that

\[
\|g_q\|^2 \geq \epsilon
\]

Applying Cauchy Schwartz inequality on equation, yields

\[
d_q = -H_q g_q + \eta \left( 1 - \rho_{q}^{\text{UAM}} d_{k-1} \right) = -H_q g_q - \eta s_q + \eta \rho_{q}^{\text{UAM}} d_{k-1}
\]

\[
\|d_q\|^2 \leq \|H_q g_q\| - \eta\|s_q\| + \|\rho_{q}^{\text{UAM}} d_{k-1}\| \quad \text{(17)}
\]

Divide both sides by \( g_q^2 \) to get

\[
\|d_q\|^2 \leq \frac{\|H_q g_q\|}{\|g_q\|^2} - \frac{\eta}{\|g_q\|^2} + \frac{\|\rho_{q}^{\text{UAM}} d_{k-1}\|}{\|g_q\|^2} \quad \text{(18)}
\]

Substitute \( \rho_{q}^{\text{UAM}} = \frac{\epsilon}{\|g_q\|^2} \), then

\[
\|s_q\|^2 \leq \frac{\|H_q g_q\|}{\|g_q\|^2} - \frac{\eta}{\|g_q\|^2} + \frac{\eta \rho_{q}^{\text{UAM}} d_{k-1}}{\|g_q\|^2}
\]
Since $\|H_q\|$ is bounded, then there exists a positive constant $\mu > 0$ such that $\|H_q\| \leq \mu$. Hence, the equation become

$$\frac{\|d_q\|^2}{\|g_q\|^2} \leq \frac{\mu}{\|H_q\|} \frac{\|p\|^2 + \|\eta g_{q-1}\|^2}{\|g_q\|^2}$$

(19)

Note that $d_0 = -H_0g_0$, then for all $q$ can summarize as

$$\frac{\|d_q\|^2}{\|g_q\|^2} \leq \sum_{p=1}^{q} \eta^q p \left( \frac{\mu + \eta}{\eta^p} \right) + \frac{\eta^q \mu}{\eta^p}$$

Substitute (16) into the equation and gives

$$\frac{\|d_q\|^2}{\|g_q\|^2} \leq \left( \frac{\mu + \eta}{\eta^p} \right) \sum_{p=1}^{q} \eta^q p \left( \frac{\mu + \eta}{\eta^p} \right) + \frac{\eta^q \mu}{\eta^p}$$

(20)

Note that, $\sum_{p=0}^{q-1} \eta^p = \frac{1 - \eta^q}{1 - \eta}$. Then,

$$\frac{\|d_q\|^2}{\|g_q\|^2} \leq \left( \frac{\mu + \eta}{\eta^p} \right) \left( \frac{1 - \eta^q}{1 - \eta} \right) + \frac{\eta^q \mu}{\eta^p}$$

$$\frac{\|d_q\|^2}{\|g_q\|^2} \leq \frac{\mu + \eta}{\eta^p} (1 - \eta)$$

$$\|g_q\|^2 \geq \left( \frac{\eta(1 - \eta)}{\mu + \eta} \right)^2$$

$$\|d_q\|^2 \geq \left( \frac{\eta(1 - \eta)}{\mu + \eta} \right)^2$$

From theorem 2, it is implied that $\sum_{q=0}^{\infty} \frac{\|g_q\|^4}{\|d_q\|^2} \geq \infty$. This contradicts Lemma 1. Hence conclude the proof.

**IV. RESULTS AND DISCUSSION**

The test problem listed by [3] has been used to analyze the new BFGS-UAM method. Different dimension varying from 2 to 5000 variables [1]. With a total of 200 test problems, the initial point will continuously subtract from the minimum point. The stopping criteria, $\|g_q\| \leq 0$ takes place when the iterations exceeds 10,000 [14].

The numerical result of each algorithm for iteration number and CPU time are displayed in Fig. 1 and 2 respectively, indebted to performance profile brought forward by [4]. In Fig.1, BFGS-UAM has the performance dealing with test functions in order to reach optimal solution compared to others. It is because BFGS-UAM has the highest percentage of solving the test functions. This result explained by the fact that this method gives highest percentage of the test problem solved. Similarly behavior in Fig. 2, $\beta'_{UAM}$ is superior among other method. It shows that this method is less time consuming. As an overall, this method is the better in both number of iteration and CPU time.

**Table 2: Numerical Results for each methods**

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Number of Iterations</th>
<th>CPU Times</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{UAM}$</td>
<td>72497</td>
<td>3795.7656</td>
</tr>
<tr>
<td>$\beta_{WYL}$</td>
<td>124217</td>
<td>4952.048</td>
</tr>
<tr>
<td>$\beta_{FR}$</td>
<td>135161</td>
<td>4889.596</td>
</tr>
</tbody>
</table>

**Fig. 1: Performance profile (number of iterations)**
V. CONCLUSION

A new hybrid coefficient for solving unconstrained optimization problems is clearly presented. The future interest is to test new hybrid coefficient with another quasi newton formula by using the inexact line search.

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