Circular D-Distance and Path Graphs

J. Veeranjaneyulu, Varma, P. L. N.

***Abstract:** In the present study we deal with the concept of circular D-distance which is the sum of D-distance and detour D-distance. We study some properties. We also obtain some results on circular D-distance, circular D-radius, circular D-self-centered graphs etc. In the areas of network theory, communications, data mining etc finding the shortest path plays an important role. Sometimes, depending on application we may have to find the longest path or both.*

***Keywords:** D-distance, detour D-distance, circular D-distance, circular D-radius, circular D-diameter.*

1. PREFACE:

If $u, v \in V(G)$ are two points the length of the geodesic u-v path is the standard distance $d(u,v)$ between u and v and the length of the detour u-v path is called *detour distance* $D(u,v)$ as introduced by Chartrand el al (see [2]).

In [5,6], by considering the degree each vertex present in the path in addition to the length of the path the *D*-length of path was determined. Using this length, Reddy Babu and Varma (in [5]), Venkateswara Rao and Varma (in [6]) defined the concept of *D*-distance and detour *D*-distance, respectively, as the minimum and maximum *D*-path lengths, respectively.

The authors have introduced a new distance, namely, *circular D-distance* in [7]. This distance is significantly different from other distances. Similarly one can talk about the circular distance using distance and detour distance. In the present study we obtained some properties of a few graphs which are trees using the circular *D*-distance.

In logistical management the circular distance plays an important role. For example, a milk van which takes a long trip to distribute the milk from dairy to the last destination covering all the delivery points on the way during the transportation of milk from dairy located in town '*x*' to a destination located in town '*y*'. On the return trip, so as to minimize the time, fuel consumption and vehicle cost during the distribute the milk the shortest route may be selected to arrive at the dairy.

Throughout the article we consider non-trivial, finite, undirected, simple and connected graphs. For any notations and terminology we refer the book [1].

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2. CIRCULAR *D*-DISTANCE

We begin with the study the concepts 'circular distance' and 'circular *D*-distance'. For reader convenience we recall some previous definitions.

Definition 2.1 (See [5]) **:** If $u, v \in V(G)$ and S is a path between them, then the *D-length* of the path, S, is determined as

$$l^D(S) = l(S) + deg(u) + deg(v) + \sum deg(w)$$

Definition 2.2 (See [5], [6]): If u,v are vertices of a connected graph *G* the D- *distance* between them is defined as $d^D(u,v) = min\{l^D(S)\}$ where the minimum is over all $u-v$ paths if *u* and *v* are distinct and $d^D(u,v) = 0$ if $u = v$. Similarly, the *detour D-distance*, $D^D(u,v)$ between *u*, *v* is defined as $D^D(u,v) = max\{l^D(S)\}$ if *u* and *v* are distinct and $D^D(u, v) = 0$ if *u* and *v* are identical where the maximum is taken over all *u* - *v* paths**.**

Now we study the concepts of circular distance and circular *D*-distance in a graph.

Definition 2.3: For $u, v \in V(G)$, the *circular distance* within them is taken as the sum of distance $d(u,v)$ and detour distance $D(u,v)$. Here $d(u,v)$ and $D(u,v)$ are the lengths of geodesic and the detour paths between *u* and *v*. This distance is denoted by $cir(u,v)$.

Definition 2.4: For $u, v \in V(G)$, the *circular D-distance*, $cir^D(u,v)$, within them is defined as sum of *D*-distance and detour *D*-distance, i.e.,

$$cir^D(u,v) = \begin{cases} min\{l^D(S)\} + max\{l^D(S)\} & if\ u \neq v \\ 0 & if\ u = v \end{cases}$$

where we consider all *u* – *v* paths while taking the minimum and maximum.

Then immediately, we have the following

Theorem 2.7: We have $cir(u,v) \le cir^D(u,v)$ For $u, v \in V(G)$.

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Proof. In a graph *G*, we know that, for any two vertices $d(u,v) \le d^D(u,v)$ and $D(u,v) \le D^D(u,v)$

From this the theorem follows immediately.

3. SOME RESULTS ON CIRCULAR *D*-DISTANCE

Some properties of circular *D*-distance will be studied in this section. We begin with an important result.

Theorem 3.1: *In a graph* G, *the circular D-distance defines a metric on the set of all vertices of G.*

The proof will be presented in [7].

Proposition 3.2: If *u, v* $\in V(G)$ *and there is one edge between u and v* $\Leftrightarrow$

$$cir^D(u,v) - D^D(u,v) = deg(u) + deg(v) + 1.$$

Proof **:** Suppose *u* and *v* are two distinct vertices in *G* which are adjacent. Then by definition

$d(u,v)=1$ and there are no intermediate vertices between *u and v*. So $\sum deg(w)=0$. Thus

$d^D(u,v) = deg(u) + deg(v) + 1$. Hence by definition of circular *D*-distance we have

$$cir^D(u,v) = d^D(u,v) + D^D(u,v)$$
$$= deg(u) + deg(v) + 1 + D^D(u,v)$$

Therefore

$$cir^D(u,v) - D^D(u,v) = deg(u) + deg(v) + 1$$

Conversely if $cir^D(u,v) = deg(u) + deg(v) + 1 + D^D(u,v)$, then by definition of circular *D*-distance we get $d(u,v) + \sum deg(w) = 1.$ Hence $d(u,v)=1$ and $\sum deg(w) = 0.$ This implies *u* and *v* are adjacent as required.

Corollary 3.3: If *u, v* $\in V(G)$ *and there is one edge between u and v* $\Leftrightarrow$

$$cir^D(u,v) = 2deg(u) + 2deg(v) + 2.$$

*Proof***:** There exist unique path within any two vertices and hence *D*-distance and detour *D*-distance are same, i.e., $d^D(u,v) = D^D(u,v)$ for $u,v \in V(G)$. Then by the definition of circular *D*–distance $cir^D(u,v) = d^D(u,v) + D^D(u,v) = 2d^D(u,v)$. Hence $cir^D(u,v)$ is equal to $2\,deg(u) + 2\,deg(v) + 2.$

Next we introduce some terminology.

Definition 3.4: The maximum circular *D*-distance to any other vertex from *u* is determined as the *circular D-eccentricity* of a vertex *u and* is represented *by* $e_C^D(u)$.

Thus $e_C^D(u) = max\{cir^D(v,u) : v \in V(G)\}.$

Definition 3.5: A vertex *u* for which $cir^D(u,v) = e_C^D(v)$ is called *circular D–eccentric vertex* of *v*. If *v* is the circular *D*-eccentric vertex of some vertex then it is called be *circular D-eccentric* vertex of *G.*

Definition 3.6: The *circular D-radius* of *G* is $r_C^D(G) = min\{e_C^D(v) : v \in V(G)\}$

Definition 3.7: The *circular D-diameter* of *G* is $d_C^D(G) = max\{e_C^D(v), v \in V(G)\}$.

Definition 3.8: In natural way using the distance we can define the concept of circular *D*-center containing of minimum *D*-centers of vertices of a graph *G* and is denoted by $c_C^D(G)$.

For all these vertices eccentricity is same as radius.

Definition 3.9: If circular *D*-center is the same of the graph *G* is called *circular D-self-centered*. Alternatively, if circular *D*-radius and circular *D*-diameter are same then graph *G* is circular *D*-self-centered.

Definition 3.10: In a graph *G* and the *circular D-periphery* is determined as

$$P_C^D(G) = \{v \in V(G) : e_C^D(v) = d_C^D(G)\}$$

Next we study a result which deals with relation between circular *D*-radius and circular *D*-diameter of a graph.

Proposition 3.11: For the graph *G*, we have $r_C^D(G) \le d_C^D(G) \le 2r_C^D(G)$.

*Proof***:** From the definition the lower limit is clear. The upper limit follows from the triangular inequality as shown below.

Suppose $u,v \in V(G)$ such that circular *D*-distance between *u* and *v* is equal to circular *D*-diameter of *G*. Let $w \in V(G)$ such that $e_C^D(w) = r_C^D(G)$. Then

$$d_C^D(G) = cir^D(u,v) \le cir^D(u,w) + cir^D(w,v)$$
$$\le e_C^D(w) + e_C^D(w) = 2e_C^D(w) = 2r_C^D(G).$$

Therefore $d_C^D(G) \le 2\, r_C^D(G)$.

Hence $r_C^D(G) \le d_C^D(G) \le 2\, r_C^D(G)$.

Remark 3.12: For the path graph, P_2, $r_C^D(P_2) = d_C^D(P_2) = 6$, as shown below in the theorem3.13.

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Theorem 3.13: Let n be the number of vertices in path graph. Then we have, for $n \geq 3$,

$$r_C^D(P_n)=\begin{cases} 3n\text{-}1 & if\ n\ is\ odd \\ 3n{+}2 & if\ n\ is\ even \end{cases} \quad and \quad d_C^D(P_n) = 6n - 6$$

and for $n=2,\ r_C^D(P_2)=d_C^D(P_2)=6.$

Proof: Let the graph P_n be contains n number of vertices so that the edges are $\{v_i v_{i+1}\}$, where i is taken as $1,2,3,\cdots,n-1.$ There exist unique path within two vertices $v_i\ and\ v_{i+1}$ in a path graph and hence $d^D(v_i,v_{i+1})=D^D(v_i,v_{i+1}).$

In P_n, the degree of each pendent vertex is one and remaining vertices of degree is 2.

Now for n=2, $e_C^D(v)=6$, where v is any vertex in the path graph P_n. Thus the circular D– radius is 6 and circular D–diameter of P_2 is 6. Hence P_2 is a self–centered graph with $r_C^D(P_n)=d_C^D(P_n)=6.$

Now let us calculate the radius and diameter of P_n for $n \geq 3$ using circular D–distance. This we will do in two cases, namely, when n is odd and even.

First case: If number of vertices (n) are even.

In the following table we calculate the circular D-distances between vertices of P_n.

	v_1	v_2	v_3	$\cdots$	$v_{\frac{n}{2}-1}$	$v_{\frac{n}{2}}$	$v_{\frac{n}{2}+1}$	$v_{\frac{n}{2}+2}$	$\cdots$	v_{n-1}	v_n
v_1	0	8	14	$\cdots$	$3n-10$	$3n-4$	$3n+2$	$3n+8$	$\cdots$	$6n-10$	$6n-6$
v_2	8	0	10	$\cdots$	$3n-14$	$3n-8$	$3n-2$	$3n+4$	$\cdots$	$6n-14$	$6n-10$
v_3	14	10	0	$\cdots$	$3n-20$	$3n-14$	$3n-8$	$3n-2$	$\cdots$	$6n-20$	$6n-14$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v_{\frac{n}{2}-1}$	$3n-10$	$3n-14$	$3n-20$	$\cdots$	0	10	16	22	$\cdots$	$3n+4$	$3n+8$
$v_{\frac{n}{2}}$	$3n-4$	$3n-8$	$3n-14$	$\cdots$	10	0	10	16	$\cdots$	$3n-2$	$3n+2$
$v_{\frac{n}{2}+1}$	$3n+2$	$3n-2$	$3n-8$	$\cdots$	16	10	0	10	$\cdots$	$3n-8$	$3n-4$
$v_{\frac{n}{2}+2}$	$3n+8$	$3n+4$	$3n-2$	$\cdots$	22	16	10	0	$\cdots$	$3n-14$	$3n-10$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{n-1}	$6n-10$	$6n-14$	$6n-20$	$\cdots$	$3n+4$	$3n-2$	$3n-8$	$3n-14$	$\cdots$	0	8
v_n	$6n-6$	$6n-10$	$6n-14$	$\cdots$	$3n+8$	$3n+2$	$3n-4$	$3n-10$	$\cdots$	8	0

Table 1. circular *D*-distances of path graphs when *n* is even

From the above table, we see that the circular D-eccentricities of vertices are $\{6n-6,\ 6n-10,\ 6n-14,\ .....3n{+}8,\ 3n{+}2,\ 3n{+}2,\ 3n{+}8,\ ....6n\text{-}10,\ 6n\text{-}6\}$.

Hence the maximum circular D-eccentricity is $6n-6,$ minimum is $3n{+}2.$ Therefore, when n is even, $r_C^D(P_n)$ is $3n{+}2$ and the $d_C^D(P_n)$ is $6n-6$.

Second case: If number of vertices (n) are odd

In the following table we calculate the circular D-distances between vertices of P_n.

	v_1	v_2	v_3	$\cdots$	$v_{\frac{n}{2}-1}$	$v_{\frac{n}{2}}$	$v_{\frac{n}{2}+1}$	$v_{\frac{n}{2}+2}$	$\cdots$	v_{n-1}	v_n
v_1	0	8	14	$\cdots$	$3n-10$	$3n-4$	$3n+2$	$3n+8$	$\cdots$	$6n-10$	$6n-6$
v_2	8	0	10	$\cdots$	$3n-14$	$3n-8$	$3n-2$	$3n+4$	$\cdots$	$6n-14$	$6n-10$
v_3	14	10	0	$\cdots$	$3n-20$	$3n-14$	$3n-8$	$3n-2$	$\cdots$	$6n-20$	$6n-14$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v_{\frac{n}{2}-1}$	$3n-10$	$3n-14$	$3n-20$	$\cdots$	0	10	16	22	$\cdots$	$3n+4$	$3n+8$
$v_{\frac{n}{2}}$	$3n-4$	$3n-8$	$3n-14$	$\cdots$	10	0	10	16	$\cdots$	$3n-2$	$3n+2$
$v_{\frac{n}{2}+1}$	$3n+2$	$3n-2$	$3n-8$	$\cdots$	16	10	0	10	$\cdots$	$3n-8$	$3n-4$
$v_{\frac{n}{2}+2}$	$3n+8$	$3n+4$	$3n-2$	$\cdots$	22	16	10	0	$\cdots$	$3n-14$	$3n-10$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{n-1}	$6n-10$	$6n-14$	$6n-20$	$\cdots$	$3n+4$	$3n-2$	$3n-8$	$3n-14$	$\cdots$	0	8
v_n	$6n-6$	$6n-10$	$6n-14$	$\cdots$	$3n+8$	$3n+2$	$3n-4$	$3n-10$	$\cdots$	8	0

Table 2. Circular *D*-distance of path graphs when *n* is odd

From the table we can observe that the circular D-eccentricities between vertices are $\{6n-6,\ 6n-10,\ 6n-14,\cdots,3n{+}5,\ 3n\text{-}1,\ 3n{+}5,\cdots,6n\text{-}10,\ 6n\text{-}6\}$.

Hence the maximum circular D-eccentricity is $6n-6$ and minimum is $3n\text{-}1$. Therefore, when n is odd, the $r_C^D(P_n)$ is $3n\text{-}1$ and the $d_C^D(P_n)$ is $6n-6$.

Combining the two cases, we get

$$r_C^D(P_n)=\begin{cases} 3n\text{-}1 & if\ n\ is\ odd \\ 3n{+}2 & if\ n\ is\ even \end{cases} \quad and \quad d_C^D(P_n) = 6\,(n\text{-}1)\ for\ n \geq 3$$

Theorem 3.14: Let $St_{n,1}$ be the star graph with n+1 vertices. Then we have

$$r_C^D(St_{n,1})=2n{+}4,\ d_C^D(St_{n,1})=2n{+}8.$$

Proof: Suppose the vertices of $St_{n,1}$ are $\{v_0,\ v_1,\ v_2,\cdots,v_n\}$ and assume that v_0 is adjacent to all other vertices so that $deg(v_0)=n$ and the degrees of remaining vertices are 1. The circular D-distances between various vertices are as shown in table 3. Using these, we can calculate the circular D-radius and circular D-diameter.

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We have $d^D(v_0,v_i)=n+2,\ d^D(v_i,v_j)=n+4$. Hence $cir^D(v_0,v_i)=2\,d^D(v_0,v_i)=2n+4\ \ (1\le i\le n),$ and $cir^D(v_i,v_j)=2\,d^D(v_i,v_j)=2n+8\ \ for\, i,j\in\{1,2,\cdots,n\}.$

	v_0	v_1	v_2	…	v_n
v_0	0	$2n+4$	$2n+4$	…	$2n+4$
v_1	$2n+4$	0	$2n+8$	…	$2n+8$
v_2	$2n+4$	$2n+8$	0	…	$2n+8$
⋮	⋮	⋮	⋮	⋮	⋮
v_n	$2n+4$	$2n+8$	$2n+8$	…	0

Table 3: Circular *D*-distances in star graphs

From the table we can observe that the circular *D*-eccentricities are $\{2n+4,\ 2n+8\}$. Thus minimum is $2n+4$ and maximum is $2n+8$. Hence circular *D*-radius is $2n+4$ and circular *D*-diameter is $2n+8$. Hence $r_C^D(St_{n,1})=2n+4,\ d_C^D(St_{n,1})=2n\ +\ 8.$

Remark 3.15: In a star graph, $d_C^D(St_{n,1})$ - $r_C^D(St_{n,1})=4.$

4. THE RELATIONS BETWEEN VARIOUS RADII AND DIAMETERS IN PATH GRAPHS

In this section we study some relations between *D*-radius, detour *D*-radius, circular *D*-radius and *D*-diameter, detour *D*-diameter, circular *D*-diameter in path graph.

Theorem 4.1: We have $r_C^D(G)=r^D(G)+r_D^D(G)$ for any path graph *G*.

Proof: Consider $v\in V(G)$. Then for *v*, the *D*-eccentricity is equal to detour *D*-eccentricity. Hence we have

$$\begin{aligned} r^D(G)+r_D^D(G) &= \min\{e^D(v)\}+\min\{e_D^D(v)\} \\ &= \min\{e^D(v)\}+\min\{e^D(v)\} \\ &= \min\{e^D(v)+e^D(v)\} \\ &= \min\{e^D(v)+e_D^D(v)\} \\ &= \min\{e_C^D(v)\} \\ &= r_C^D(G) \end{aligned}$$

Hence $r_C^D(G)=r^D(G)+r_D^D(G).$

Theorem 4.2: : For any path graph *G* we have $d_C^D(G)=d^D(G)+d_D^D(G).$

Proof: Consider a vertex *v* of *G*. Then, for *v*, *D*-eccentricity is equal to detour *D*-eccentricity. Thus

$$\begin{aligned} d^D(G)+d_D^D(G) &= \max\{e^D(v)\}+\max\{e_D^D(v)\} \\ &= \max\{e^D(v)\}+\max\{e^D(v)\} \\ &= \max\{e^D(v)+e^D(v)\} \\ &= \max\{e^D(v)+e_D^D(v)\} \\ &= d_C^D(G). \end{aligned}$$

Hence $d_C^D(G)=d^D(G)+d_D^D(G).$

From the above theorems we can derive the following

Conclusion 4.3: For any path graph *G* we have $r_C^D(G)=2r^D(G)$ and $r_C^D(G)=2r_D^D(G)$.

In a path graph, we have $r^D(G)=r_D^D(G).$ Then by theorem 4.1, we have

$$\begin{aligned} r_C^D(G) &= r^D(G)+r_D^D(G) \\ &= r^D(G)+r^D(G) \\ &= 2\,r^D(G) \\ r_C^D(G) &= 2\,r_D^D(G). \end{aligned}$$

Similarly we can write

Conclusion 4.4: In a path graph *G* we have $d_C^D(G)=2d^D(G)$ and $d_C^D(G)=2d_D^D(G)$

In a path graph, we have $d^D(G)=d_D^D(G).$ Then from the theorem 4.2 we have

$$\begin{aligned} d_C^D(G) &= d^D(G)+d_C^D(G) \\ &= d^D(G)+d^D(G) \\ &= 2\,d^D(G) \end{aligned}$$

Similarly, we can write $d_C^D(G)=2\,d_D^D(G).$

Conclusion 4.5: In a path graph *G* we have $r_D^D(G)\le d_D^D(G)\le r_C^D(G)$ and $r^D(G)\le d^D(G)\le r_C^D(G).$

From the relation (see [6]) $r_D^D(G)\le d_D^D(G)\le 2\,r_D^D(G)$ we have using conclusion 4.3 $r_D^D(G)\le d_D^D(G)\le r_C^D(G)$.
Similarly, from the relation (see [5]) we have $r^D(G)\le d^D(G)\le 2\,r^D(G)$. Using the above conclusion 4.3, from this we get $r^D(G)\le d^D(G)\le r_C^D(G)$.

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REFERENCES:

1. Buckley, F., Harary, F., *Distance in graphs*, Addison-Wesley, Longman, 1990.
2. Chartrand, G., Escuadro, H., Zhang, P., *Detour distance in graphs*, J. Combin. Comput 53 (2005), 75-94.
3. Chartrand, G., Zhang, P., *Distance in graphs-taking the long view*, *AKCE J. Graphs Combin.*, 1 (2004), 1-13.
4. Chartrand, G and P. Zang, *Introduction to Graph Theory*, Tata McGraw-Hill, (2006).
5. Reddy Babu, D., Varma, P.L.N., $D-$ *distance in graphs*, Gold. Res. Thoughts, 2 (2013) 53-58.
6. Venkateswara Rao V., Varma P. L. N., *Detour Distance in Graphs w.r.t. D-distance*, Ponte International Journal of Sciences and Research. Res.73 (7) (2017) 19-28.
7. Veeranjaneyulu J, P.L.N. Varma, *Circular D-distance in graphs*, Preprint, 2018.