Properties of Hypergeometric Functions for Functions Certain Subclasses of Sakaguchi Type Functions

N. P. Damodaran, B. Srutha Keerthi

Abstract: The purpose of the present paper is to give a sufficient condition for [Gaussian] hyper geometric function to be in a subclass of analytic function which is also necessary condition under additional restrictions. Similar results for the corresponding subclasses of analytic functions are also obtained. Furthermore an integral operator related to the hypergeometric function is determined.2000 Subject Classification: 30C45.

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I. INTRODUCTION

Let A be the class consisting of functions of the form

$$u(z) = z + \sum_{\eta=2}^{\infty} a_\eta z^\eta \quad (1)$$

that are analytic in the open unit disk D = {z : |z| < 1}.

$$\mathcal{R}\left\{\frac{zu'(z)}{u(z)} - \gamma\right\} \geq \left|\frac{zu'(z)}{u(z)} - 1\right|, \quad (\gamma \in \mathcal{R}, z \in D),$$

and u(z) ∈ UCV(γ), the class of uniformly convex functions of order γ, if and only if $zu'(z)$ ∈ $S_p(\gamma)$. Motivated by work of Srutha Keerthi and et.al[13], we define following classes.

respectively. For convenience, we write $S^*(0) = S^*$ and K(0) = K (see, e.g., Srivastava and Owa [12]).

Motivated by geometric considerations, Goodman [4, 5] introduced the class UCV and UST of uniformly convex and starlike functions. Ronning [8] (also, see [6]) gave a more applicable one variable analytic characterization for UCV. That is, a function u(z) of the form (1) is in UCV if and only if

Let S, $S^*(\alpha)$ and K(α) denote the subclass of A consisting of univalent, starlike and convex functions of order α,

$$\mathcal{R}\left\{1 + \frac{zu''(z)}{u'(z)}\right\} \geq \left|\frac{zu''(z)}{u'(z)}\right|, \quad (z \in D).$$

We note [4] that the classical Alexander's result, u(z) ∈ K ⇔ $zu'(z)$ ∈ S^* does not hold between the classes UCV and UST. Later on Ronning [9] introduced the class S_p consisting of functions such that u(z) ∈ UCV ⇔ $zu'(z)$ ∈ S_p. And also in [8], Ronning generalized the classes UCV and S_p by introduced a parameter γ in the following way i.e., A function u(z) of the form (1) is in

$S_p(\gamma)$ if it satisfies the analytic characterization

Definition 1.1. If the function u(z) of the form (1) which stated in the subclass B (ρ, γ,t), it satisfies the holomorphic characterization

$$\mathcal{R}\left[\frac{(1-t)\rho z^3u'''(z) + (1+2\rho)z^2u''(z) + zu'(z)}{\rho z^2[u''(z) - t^2(u''(tz))] + z[u'(z) - t(u'(tz))]} - \gamma\right] \geq \left|\frac{(1-t)\rho z^3u'''(z) + (1+2\rho)z^2u''(z) + zu'(z)}{\rho z^2[u''(z) - t^2(u''(tz))] + z[u'(z) - t(u'(tz))]} - 1\right| \quad (2)$$

(γ ∈ R, 0 ≤ ρ ≤ 1, $z \in D$,|t|≤1 and t≠1).

We also note that B(0, γ,0) = UCV(γ).

For case of ρ=0, we get the following class.

Definition 1.2. If the function u(z) of the form (1) which stated in the Subclass C (γ,t), it satisfies the holomorphic characterization

$$\mathcal{R}\left[\frac{(1-t)[z^2u''(z) + zu'(z)]}{z[u'(z) - t(u'(tz))]} - \gamma\right] \geq \left|\frac{(1-t)[z^2u''(z) + zu'(z)]}{z[u'(z) - t(u'(tz))]} - 1\right| \quad (3)$$

(γ ∈ R, $z \in D$,|t|≤1 and t≠1).

We defined T is a subclass of S consisting of functions of the form

$$u(z) = z - \sum_{\eta=2}^{\infty} a_\eta z^\eta, \quad (a_\eta \geq 0) \quad (4)$$

and let $B_T(\rho, \gamma, t) = B(\rho, \gamma, t) \cap T$.

Let F(p, q; r; z) be the (Gaussian) hypergeometric function defined by

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$$F\left(p,\ q,\ r,\ z\right)=\sum_{\eta=0}^{\infty}\frac{(p)_{\eta}(q)_{\eta}}{(r)_{\eta}(1)_{\eta}}z^{\eta},$$

where $\eta \neq 0, -1, -2, \ldots$ and $(\beta)_{\eta}$ is the Pochhammer symbol defined by

$$(\beta)_{\eta}=\begin{cases}1 & \text{if } \eta=0\\ \beta(\beta+1)\ldots(\beta+\eta-1) & \text{if } \eta\in\mathbb{N}=\{1,2,\ldots\}.\end{cases}$$

We note that $F(p,q;r;1)$ converges for $R(p-q-r)>0$ and related to the Gamma functions by

$$F(p,q;r;1)=\frac{\Gamma(r)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}. \quad (5)$$

Merkes and Scott [7] and Ruscheweyh and Singh [10] used continued fractions to find sufficient conditions for zF(p, q; r; 1) to be in $S^{*}(\gamma)$ $(0 \leq \gamma < 1)$ for various choices of parameters p, q and r. Carlson and Shaffer [2] showed how some convolution results about $S^{*}(\gamma)$ may be expressed in terms of a linear operator acting on hypergeometric functions. Recently, Silverman [11] gave necessary and sufficient condition for zF(p, q; r; z) to be in $S^{*}(\gamma)$ and K(γ).

II. CONDITIONS FOR STARLIKENESS AND UNIFORM CONVEXITY

Lemma 2.1. A sufficient condition for power series form u(z) which stated to be the Subclass B(ρ, γ,t) is that

$$\sum_{k=2}^{\infty}\Big[2\rho\eta(\eta-1)(\eta-2)+[(2+4\rho)-\rho(1+t+t^{2}u_{\eta})(1+\gamma)]\eta(\eta-1)$$
$$+[(1-\gamma)-tu_{\eta}(1+\gamma)](\eta-1)+[(1-\gamma)-tu_{\eta}(1+\gamma)]\Big]\left|a_{\eta}\right|\leq(1-\gamma)-t(1+\gamma). \quad (6)$$

Proof. In view of the definition of B(ρ, γ,t) it is sufficient if we verify the condition

$$\left|\frac{(1-t)\rho z^{3}u'''(z)+(1+2\rho)z^{2}u''(z)+zu'(z)}{\rho z^{2}[u''(z)-t^{2}(u''(tz))]+z[u'(z)-t(u'(tz))]}-1\right|$$
$$\leq\mathcal{R}\left[\frac{(1-t)\rho z^{3}u'''(z)+(1+2\rho)z^{2}u''(z)+zu'(z)}{\rho z^{2}[u''(z)-t^{2}(u''(tz))]+z[u'(z)-t(u'(tz))]}-1\right]+1-\gamma.$$

We have

$$\left|\frac{(1-t)\rho z^{3}u'''(z)+(1+2\rho)z^{2}u''(z)+zu'(z)}{\rho z^{2}[u''(z)-t^{2}(u''(tz))]+z[u'(z)-t(u'(tz))]}-1\right|$$
$$\leq R\left[\frac{(1-t)\rho z^{3}u'''(z)+(1+2\rho)z^{2}u''(z)+zu'(z)}{\rho z^{2}[u''(z)-t^{2}(u''(tz))]+z[u'(z)-t(u'(tz))]}-1\right]$$
$$\leq 2\left|\frac{(1-t)\rho z^{3}u'''(z)+(1+2\rho)z^{2}u''(z)+zu'(z)}{\rho z^{2}[u''(z)-t^{2}(u''(tz))]+z[u'(z)-t(u'(tz))]}-1\right|$$
$$\leq\frac{\sum_{\eta=2}^{\infty}[2\rho\eta(\eta-1)(\eta-2)+2[(1+2\rho)-\rho(1+t+t^{2}u_{\eta})]\eta(\eta-1)-2\eta u_{\eta})]\left|a_{\eta}\right|-2t}{(1-t)+\sum_{\eta=2}^{\infty}[\rho\eta(\eta-1)(1+t+t^{2}u_{\eta})+\eta(1+tu_{\eta})]\left|a_{\eta}\right|},$$

Where

$$u_{\eta}=\frac{1-t^{\eta}}{1-t},(\eta\in N).$$

The above expression is bounded by $1-\gamma$ if and only if equation (6) is satisfied and the proof is complete.

For case of $\rho=0$, we get the following.

Corollary 2.1. A sufficient condition for power series form u(z) which stated to be the Subclass C(γ,t) is that

$$\sum_{\eta=2}^{\infty}\Big[2\eta(\eta-1)+[(1-\gamma)-tu_{\eta}(1+\gamma)](\eta-1)+[(1-\gamma)-tu_{\eta}(1+\gamma)]\Big]\left|a_{\eta}\right|\leq(1-\gamma)-t(1+\gamma). \quad (7)$$

Lemma 2.2. A necessary and sufficient condition for power series form u(z) which stated to be the Subclass BT(ρ, γ,t) is that

$$\sum_{k=2}^{\infty}\Big[2\rho\eta(\eta-1)(\eta-2)+[(2+4\rho)-\rho(1+t+t^{2}u_{\eta})(1+\gamma)]\eta(\eta-1)$$
$$+[(1-\gamma)-tu_{\eta}(1+\gamma)](\eta-1)+[(1-\gamma)-tu_{\eta}(1+\gamma)]\Big]a_{\eta}\leq(1-\gamma)-t(1+\gamma). \quad (8)$$

Proof. Suppose u(z) of the form (4) to be in the class $B_{T}(\rho,\gamma,t)$. Then

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$$(1-\gamma)+\mathcal{R}\left[\frac{(1-t)\rho z^3u'''(z)+(1+2\rho)z^2u''(z)+zu'(z)}{\rho z^2[u''(z)-t^2(u''(tz))]+z[u'(z)-t(u'(tz))]}-\gamma\right]$$
$$\geq\left|\frac{(1-t)\rho z^3u'''(z)+(1+2\rho)z^2u''(z)+zu'(z)}{\rho z^2[u''(z)-t^2(u''(tz))]+z[u'(z)-t(u'(tz))]}-1\right|$$

or equivalently, for $z \in D$,

$$(1-\gamma)-\mathcal{R}\left[\frac{\sum_{\eta=2}^{\infty}[\rho\eta(\eta-1)(\eta-2)+[(1+2\rho)-\rho(1+t+t^2u_\eta)]\eta(\eta-1)-\eta tu_\eta)]\,a_\eta z^\eta-t}{z(1-t)-\sum_{\eta=2}^{\infty}[\rho\eta(\eta-1)(1+t+t^2u_\eta)+\eta(1+tu_\eta)]a_\eta z^\eta}\right]$$
$$\geq\left|\frac{\sum_{\eta=2}^{\infty}[\rho\eta(\eta-1)(\eta-2)+[(1+2\rho)-\rho(1+t+t^2u_\eta)]\eta(\eta-1)-\eta tu_\eta)]\,a_\eta z^\eta+t}{z(1-t)-\sum_{\eta=2}^{\infty}[\rho\eta(\eta-1)(1+t+t^2u_\eta)+\eta(1+tu_\eta)]a_\eta z^\eta}\right|.$$

Choosing values of z on the real axis, so that the left side of this inequality is real, and letting $z \to 1$, we obtain

$$(1-\gamma)\left[(1+t)-\sum_{\eta=2}^{\infty}[\rho\eta(\eta-1)(1+t+t^2u_\eta)+\eta(1+tu_\eta)]a_\eta\right]$$
$$-\sum_{\eta=2}^{\infty}[\rho\eta(\eta-1)(\eta-2)+[(1+2\rho)-\rho(1+t+t^2u_\eta)]\eta(\eta-1)-\eta tu_\eta)]\,a_\eta-t$$
$$\geq\sum_{\eta=2}^{\infty}[\rho\eta(\eta-1)(\eta-2)+[(1+2\rho)-\rho(1+t+t^2u_\eta)]\eta(\eta-1)-\eta tu_\eta)]\,a_\eta+t$$
$$\Rightarrow\quad\sum_{\eta=2}^{\infty}\Big[2\rho\eta(\eta-1)(\eta-2)+[(2+4\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)]\eta(\eta-1)$$
$$+[(1-\gamma)-tu_\eta(1+\gamma)](\eta-1)+[(1-\gamma)-tu_\eta(1+\gamma)]\Big]\,a_\eta\leq(1-\gamma)-t(1+\gamma).$$

Conversely, assume that (8) is true.

In order to prove that $u(z) \in B_T(\rho,\gamma,t)$ we prove that

$$\left|\frac{(1-t)\rho z^3u'''(z)+(1+2\rho)z^2u''(z)+zu'(z)}{\rho z^2[u''(z)-t^2(u''(tz))]+z[u'(z)-t(u'(tz))]}-1\right|$$
$$-\mathcal{R}\left[\frac{(1-t)\rho z^3u'''(z)+(1+2\rho)z^2u''(z)+zu'(z)}{\rho z^2[u''(z)-t^2(u''(tz))]+z[u'(z)-t(u'(tz))]}-1\right]$$
$$\leq(1-\gamma)-t(1+\gamma).$$

Thus, for $z \in D$,

$$\left|\frac{(1-t)\rho z^3u'''(z)+(1+2\rho)z^2u''(z)+zu'(z)}{\rho z^2[u''(z)-t^2(u''(tz))]+z[u'(z)-t(u'(tz))]}-1\right|$$
$$-\mathcal{R}\left[\frac{(1-t)\rho z^3u'''(z)+(1+2\rho)z^2u''(z)+zu'(z)}{\rho z^2[u''(z)-t^2(u''(tz))]+z[u'(z)-t(u'(tz))]}-1\right]$$

$$\le \left| \frac{\sum_{\eta=2}^{\infty} [\rho\eta(\eta-1)(\eta-2)+[(1+2\rho)-\rho(1+t+t^2u_\eta)]\eta(\eta-1)-\eta t u_\eta)]\, a_\eta z^\eta + t}{z(1-t)-\sum_{\eta=2}^{\infty}[\rho\eta(\eta-1)(1+t+t^2u_\eta)+\eta(1+tu_\eta)]a_\eta z^\eta} \right|$$

$$+\mathcal{R}\left[\frac{\sum_{\eta=2}^{\infty} [\rho\eta(\eta-1)(\eta-2)+[(1+2\rho)-\rho(1+t+t^2u_\eta)]\eta(\eta-1)-\eta t u_\eta)]\, a_\eta z^\eta + t}{z(1-t)-\sum_{\eta=2}^{\infty}[\rho\eta(\eta-1)(1+t+t^2u_\eta)+\eta(1+tu_\eta)]a_\eta z^\eta} \right]$$

$$\le \frac{\sum_{k=2}^{\infty} [2\rho k(k-1)(k-2)+2[(1+2\rho)-\rho(1+t+t^2u_k)]k(k-1)-2k t u_k)]\, a_k z^k + 2t}{z(1-t)-\sum_{k=2}^{\infty}[\rho k(k-1)(1+t+t^2u_k)+k(1+tu_k)]a_k z^k}$$

$$\le (1-\gamma)-t(1+\gamma).$$

as equation (8) is true.

For case of ρ=0, we get the following.

Corollary 2.2. A necessary and sufficient condition for power series form u(z) which stated to be the Subclass C(γ,t) is that

$$\sum_{\eta=2}^{\infty}\left[2\eta(\eta-1)+[(1-\gamma)-tu_\eta(1+\gamma)](\eta-1)+[(1-\gamma)-tu_\eta(1+\gamma)]\right]\mathrm{a}_\eta \le (1-\gamma)-t(1+\gamma). \tag{9}$$

Theorem 2.1. If p, q > 0 and r > p+q+3, then a sufficient condition for zF(p,q;r;z) is said to be in the subclass B(ρ, γ,t) (0 ≤ ρ ≤ 1, −1 ≤ γ < 1, |t|≤1 and t≠1) is that

$$\frac{\Gamma(\mathrm{r})\Gamma(\mathrm{r}-p-q)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-q)}\left[\frac{2\rho(\mathrm{p})_3(\mathrm{q})_3+[(2+10\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)](\mathrm{r}-p-q-3)(\mathrm{p})_2(\mathrm{q})_2}{[(1-\gamma)-tu_{\eta+2}(1+\gamma)](\mathrm{r}-p-q-3)_3}\right.$$
$$\left.\frac{[2(2+4\rho)-2\rho(1+t+t^2u_{\eta+1})(1+\gamma)+(1-\gamma)-tu_{\eta+1}(1+\gamma)](\mathrm{r}-p-q-3)_2\mathrm{pq}}{[(1-\gamma)-tu_{\eta+2}(1+\gamma)](\mathrm{r}-p-q-3)_3}+1\right] \tag{10}$$
$$\le \frac{[(1-\gamma)-t(1+\gamma)]}{[(1-\gamma)-tu_{\eta+2}(1+\gamma)]}+1.$$

Condition (10) is necessary and sufficient for F_1 defined by F_1(p, q; r; z) = z[2 − F(p, q; r; z)] to be in B_T(ρ, γ,t).

Proof. Since $\mathrm{zF(p,q;r;z)} = \mathrm{z}+\sum_{\eta=2}^{\infty}\frac{(\mathrm{p})_{\eta\text{-}1}(\mathrm{q})_{\eta\text{-}1}}{(\mathrm{r})_{\eta\text{-}1}(1)_{\eta\text{-}1}}\mathrm{z}^\eta$, according to Lemma 2.1, we need only to show that

$$\sum_{\eta=2}^{\infty}\left[2\rho\eta(\eta-1)(\eta-2)+[(2+4\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)]\eta(\eta-1)\right.$$
$$\left.+[(1-\gamma)-tu_\eta(1+\gamma)](\gamma-1)+[(1-\gamma)-tu_\eta(1+\gamma)]\right]\frac{(\mathrm{p})_{\eta\text{-}1}(\mathrm{q})_{\eta\text{-}1}}{(\mathrm{r})_{\eta\text{-}1}(1)_{\eta\text{-}1}} \le [(1-\gamma)-t(1+\gamma)].$$

Now

$$\sum_{\eta=2}^{\infty}\left[2\rho\eta(\eta-1)(\eta-2)+[(2+4\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)]\eta(\eta-1)\right.$$
$$\left.+[(1-\gamma)-tu_\eta(1+\gamma)](\eta-1)+[(1-\gamma)-tu_\eta(1+\gamma)]\right]\frac{(\mathrm{p})_{\eta\text{-}1}(\mathrm{q})_{\eta\text{-}1}}{(\mathrm{r})_{\eta\text{-}1}(1)_{\eta\text{-}1}}$$
$$=\sum_{\eta=1}^{\infty}\left[2\rho\eta(\eta+1)(\eta-1)+[(2+4\rho)-\rho(1+t+t^2u_{\eta+1})(1+\gamma)]\eta(\eta+1)\right.$$
$$\left.+[(1-\gamma)-tu_{\eta+1}(1+\gamma)]\eta+[(1-\gamma)-tu_{\eta+1}(1+\gamma)]\right]\frac{(\mathrm{p})_\eta(\mathrm{q})_\eta}{(\mathrm{r})_\eta(1)_\eta}. \tag{11}$$

Noting $(\beta)_\eta = \beta(\beta+1)_{\eta+1}$ and then applying (4), we may express equation (11) as

$$\frac{2\rho(\mathrm{p})_3(\mathrm{q})_3}{(\mathrm{r})_3}\sum_{\eta=3}^{\infty}\frac{(\mathrm{p}+3)_{\eta-3}(\mathrm{q}+3)_{\eta-3}}{(\mathrm{r}+3)_{\eta-3}(1)_{\eta-3}}$$
$$+\frac{[(2+10\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)](\mathrm{p})_2(\mathrm{q})_2}{(\mathrm{r})_2}\sum_{\eta=2}^{\infty}\frac{(\mathrm{p}+2)_{\eta-2}(\mathrm{q}+2)_{\eta-2}}{(\mathrm{r}+2)_{\eta-2}(1)_{\eta-2}}$$
$$+\frac{[2(2+4\rho)-2\rho(1+t+t^2u_{\eta+1})(1+\gamma)+(1-\gamma)-tu_{\eta+1}(1+\gamma)]\mathrm{pq}}{\mathrm{c}}\sum_{\eta=1}^{\infty}\frac{(\mathrm{p}+1)_{\eta-1}(\mathrm{q}+1)_{\eta-1}}{(\mathrm{r}+1)_{\eta-1}(1)_{\eta-1}}$$
$$+[(1-\gamma)-tu_{\eta+1}(1+\gamma)]\sum_{\eta=1}^{\infty}\frac{(\mathrm{p})_\eta(\mathrm{q})_\eta}{(\mathrm{r})_\eta(1)_\eta}$$

$$=\frac{2\rho(\mathrm{p})_3(\mathrm{q})_3}{(\mathrm{r})_3}\frac{\Gamma(\mathrm{r}+3)\Gamma(\mathrm{r}-p-q-3)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-q)}$$
$$+\frac{[(2+10\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)](\mathrm{p})_2(\mathrm{q})_2}{(\mathrm{r})_2}\frac{\Gamma(\mathrm{r}+2)\Gamma(\mathrm{r}-p-q-2)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-p)}$$
$$+\frac{[2(2+4\rho)-2\rho(1+t+t^2u_{\eta+1})(1+\gamma)+(1-\gamma)-tu_{\eta+1}(1+\gamma)]\mathrm{pq}}{r}\frac{\Gamma(\mathrm{r}+1)\Gamma(\mathrm{r}-p-q-1)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-p)}$$
$$+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\frac{\Gamma(\mathrm{r})\Gamma(\mathrm{r}-p-q)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-q)}-[(1-\gamma)-tu_{\eta+2}(1+\gamma)].$$

$$=\frac{\Gamma(\mathrm{r})\Gamma(\mathrm{r}-p-q)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-q)}\left[\frac{2\rho(\mathrm{p})_3(\mathrm{q})_3+[(2+10\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)](\mathrm{r}-p-q-3)(\mathrm{p})_2(\mathrm{q})_2}{(\mathrm{r}-p-q-3)_3}\right.$$
$$\left.+\frac{[2(2+4\rho)-2\rho(1+t+t^2u_{\eta+1})(1+\gamma)+(1-\gamma)-tu_{\eta+1}(1+\gamma)](\mathrm{r}-p-q-3)_2pq}{(\mathrm{r}-p-q-3)_3}+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\right]$$
$$-[(1-\gamma)-tu_{\eta+2}(1+\gamma)].$$

But this last expression is bounded above by (1 – γ) if and only if equation (11) holds, and hence the proof.

Since $\mathrm{F}_1(\mathrm{p,q;r;z})=\mathrm{z}-\sum_{\eta=2}^{\infty}\frac{(\mathrm{p})_{\eta-1}(\mathrm{q})_{\eta-1}}{(\mathrm{c})_{\eta-1}(1)_{\eta-1}}\mathrm{z}^\eta$, the necessity of (6) for F_1 to be in B_T(ρ,γ ,t) follows from Lemma 2.2.

For the Case ρ=0, we get following.

Corollary 2.3. If p, q > 0 and r > p+q+2, then a sufficient condition for zF(p,q;r;z) is said to in the Subclass C(γ,t) (–1 ≤ γ < 1, |t|≤1 and t≠1) is that

$$\frac{\Gamma(\mathrm{r})\Gamma(\mathrm{r}-p-q)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-q)}\left[\frac{2(\mathrm{p})_2(\mathrm{q})_2}{[(1-\gamma)-tu_{\eta+2}(1+\gamma)](\mathrm{r}-p-q-1)(\mathrm{r}-p-q-2)}\right.$$
$$\left.+\frac{[(5-\gamma)-tu_{\eta+1}(1+\gamma)]\mathrm{pq}}{[(1-\gamma)-tu_{\eta+2}(1+\gamma)](\mathrm{r}-p-q-1)}+1\right]\le\frac{[(1+\gamma)-t(1+\gamma)]}{[(1-\gamma)-tu_{\eta+2}(1+\gamma)]}+1. \qquad (12)$$

which was proved by Srutha Keerthi and et.al[13]

Theorem 2.2. If p, q > –1, pq < 0 and r> p+q+3, then a necessary and sufficient condition for zF(p, q; r; z) which stated to be the Subclass B_T(ρ,γ,t) (0 ≤ ρ ≤ 1, –1 ≤ γ < 1, |t|≤1 and t≠1) is that

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$$\frac{\Gamma(r+1)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-p)}\Bigg[\frac{2\rho(p+1)_2(q+1)_2}{(r-p-q-3)_3}+\frac{[(2+10\rho)-\rho(1+t+t^2u_{\eta+1})(1+\gamma)](p+1)(q+1)}{(r-p-q-2)_2}$$
$$\frac{[2(2+4\rho)-2\rho(1+t+t^2u_{\eta+2})(1+\gamma)+(1-\gamma)-tu_{\eta+2}(1+\gamma)]}{(p-q-r-1)}+\frac{(1-\gamma)-tu_{\eta+2}(1+\gamma)]}{pq}\Bigg]$$
$$\leq[(1-\gamma)-t(1+\gamma)]\frac{r}{|pq|}+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\frac{r}{pq}. \quad (13)$$

Proof. Since

$$zF(p,q;r;z)=z+\frac{pq}{r}\sum_{\eta=2}^{\infty}\frac{(p+1)_{\eta-2}(q+1)_{\eta-2}}{(r+1)_{\eta-2}(1)_{\eta-1}}z^{\eta}$$
$$=z-\left|\frac{pq}{r}\right|\sum_{\eta=2}^{\infty}\frac{(p+1)_{\eta-2}(q+1)_{\eta-2}}{(q+1)_{\eta-2}(1)_{\eta-1}}z^{\eta}, \quad (14)$$

according to Lemma 2.2, we must show that

$$\sum_{\eta=2}^{\infty}\Big[2\rho\eta(\eta-1)(\eta-2)+[(2+4\rho)-\rho(1+t+t^2u_{\eta})(1+\gamma)]\eta(\eta-1)$$
$$+[(1-\gamma)-tu_{\eta}(1+\gamma)](\eta-1)+[(1-\gamma)-tu_{\eta}(1+\gamma)]\Big]\frac{(p+1)_{\eta-2}(q+1)_{\eta-2}}{(r+1)_{\eta-2}(1)_{\eta-1}}$$
$$\leq\left|\frac{r}{pq}\right|[(1-\gamma)-t(1+\gamma)]. \quad (15)$$

Now

$$\sum_{\eta=0}^{\infty}\Big[2\rho\eta(\eta+1)(\eta+2)+[(2+4\rho)-\rho(1+t+t^2u_{\eta+2})(1+\gamma)](\eta+1)(\eta+2)$$
$$+[(1-\gamma)-tu_{\eta+2}(1+\gamma)](k-1)+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\Big]\frac{(p+1)_{\eta}(q+1)_{\eta}}{(r+1)_{\eta}(1)_{\eta+1}}$$
$$=2\rho\sum_{\eta=2}^{\infty}\frac{(p+1)_{\eta}(q+1)_{\eta}}{(r+1)_{\eta}(1)_{\eta-2}}+[(2+10\rho)-\rho(1+t+t^2u_{\eta+1})(1+\gamma)]\sum_{\eta=1}^{\infty}\frac{(p+1)_{\eta}(q+1)_{\eta}}{(r+1)_{\eta}(1)_{\eta-1}}$$
$$+[2(2+4\rho)-2\rho(1+t+t^2u_{\eta+2})(1+\gamma)+(1-\gamma)-tu_{\eta+2}(1+\gamma)]\sum_{\eta=0}^{\infty}\frac{(p+1)_{\eta}(q+1)_{\eta}}{(r+1)_{\eta}(1)_{\eta}}$$
$$+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\sum_{\eta=0}^{\infty}\frac{(p+1)_{\eta}(q+1)_{\eta}}{(r+1)_{\eta}(1)_{\eta+1}}$$
$$=\frac{\Gamma(r+1)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}\Bigg[\frac{2\rho(p+1)_2(q+1)_2}{(r-p-q-3)_3}+\frac{[(2+10\rho)-\rho(1+t+t^2u_{k+1})(1+\gamma)](p+1)(q+1)}{(r-p-q-2)_2}$$
$$\frac{[2(2+4\rho)-2\rho(1+t+t^2u_{\eta+2})(1+\gamma)+(1-\gamma)-tu_{\eta+2}(1+\gamma)]}{(r-p-q-1)}+\frac{(1-\gamma)-tu_{\eta+2}(1+\gamma)]}{pq}\Bigg]$$
$$\leq[(1-\gamma)-t(1+\gamma)]\frac{r}{|pq|}+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\frac{r}{pq}.$$

For the Case ρ=0, we get following.

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Corollary 2.4 If p, q > –1, pq < 0 and r > p+q+2, a necessary and sufficient condition for zF(p, q; r; z) which state to be the Subclass C_T (γ,t) (–1 ≤ γ < 1, |t≤1 and t≠1|) is that

$$\frac{\Gamma(\mathrm{r}+1)\Gamma(\mathrm{r}-p-q)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-q)}\left[\frac{[(5-\gamma)-tu_{\eta+2}(1+\gamma)]\mathrm{pq}}{(\mathrm{r}-p-q-1)}+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\frac{r}{pq}\right]$$
$$\leq[(1-\gamma)-t(1+\gamma)]\frac{r}{|pq|}+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\frac{r}{pq}, \tag{16}$$

which was proved by Srutha Keerthi and et.al[13].

III. AN INTEGRAL OPERATOR

A particular integral operator G(p, q; r; z) acting on F(p, q; r; z) as follows:

$$\mathrm{G(p,q;r;z)}=\int_0^{\mathrm{z}}\mathrm{F(p,q;r;t)dt}. \tag{17}$$

Theorem 3.1.

(i) If p, q> 1 and r > p+q+2, a sufficient condition for G(p, q; r; z) defined by (17) is said to in the Subclass B(ρ, γ, t) is that

$$\frac{\Gamma(\mathrm{r})\Gamma(\mathrm{r}-p-q)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-q)}\left[\frac{2\rho(\mathrm{p})_2(\mathrm{q})_2}{(\mathrm{r}-p-q-2)_2}+\frac{[(2+4\rho)-\rho(1+t+t^2u_{\eta+2})(1+\gamma)]pq}{(\mathrm{r}-p-q-1)}+(1-\gamma)-tu_{\eta+2}(1+\gamma)]\right.$$

$$\leq[(1-\gamma)-t(1+\gamma)]+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]. \tag{18}$$

(ii) If p, q > –1, pq < 0 and r > max{0, p+q+2}, then G(p, q; r; z) defined by (17) is said to in the Subclass B_T(ρ, γ, t) if and only if

$$\frac{\Gamma(\mathrm{r}+1)\Gamma(\mathrm{r}-p-q)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-q)}\left[\frac{2\rho(p+1)(q+1)}{(\mathrm{r}-p-q-1)(\mathrm{r}-p-q-2)}\right.$$
$$\left.+\frac{[(2+4\rho)-\rho(1+t+t^2u_{\eta+2})(1+\gamma)]}{(\mathrm{r}-p-q-1)}+\frac{(1-\gamma)-tu_{\eta+2}(1+\gamma)]}{pq}\right]$$
$$\leq[(1-\gamma)-t(1+\gamma)]\frac{r}{|pq|}+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\frac{r}{pq}. \tag{19}$$

Proof. Since

$$\mathrm{G(p,q;r;z)}=\mathrm{z}+\sum_{\eta=2}^{\infty}\frac{(\mathrm{p})_{\eta-1}(\mathrm{q})_{\eta-1}}{(\mathrm{r})_{\eta-1}(1)_{\eta}},$$

we note that

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$$\sum_{\eta=2}^{\infty}\Big[2\rho\eta(\eta-1)(\eta-2)+[(2+4\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)]\eta(\eta-1)$$
$$+[(1-\gamma)-tu_\eta(1+\gamma)](\eta-1)+[(1-\gamma)-tu_\eta(1+\gamma)]\Big]\frac{(\mathrm{p})_{\eta-1}(\mathrm{q})_{\eta-1}}{(\mathrm{r})_{\eta-1}(1)_\eta}$$
$$=\sum_{\eta=1}^{\infty}\Big[2\rho(\eta+1)\eta(\eta-1)+[(2+4\rho)-\rho(1+t+t^2u_{\eta+1})(1+\gamma)](\eta+1)\eta$$
$$+[(1-\gamma)-tu_{\eta+1}(1+\gamma)](\eta-1)+[(1-\gamma)-tu_{\eta+1}(1+\gamma)]\Big]\frac{(\mathrm{p})_\eta(\mathrm{q})_\eta}{(\mathrm{r})_\eta(1)_{\eta+1}}$$

$$=2\rho\sum_{\eta=2}^{\infty}\frac{(\mathrm{p})_\eta(\mathrm{q})_\eta}{(\mathrm{r})_\eta(1)_{\eta-2}}+[(2+4\rho)-\rho(1+t+t^2u_{\eta+1})(1+\gamma)]\sum_{\eta=1}^{\infty}\frac{(\mathrm{p})_\eta(\mathrm{q})_\eta}{(\mathrm{r})_\eta(1)_{\eta-1}}$$
$$+[(1-\gamma)-tu_{\eta+1}(1+\gamma)]\sum_{\eta=1}^{\infty}\frac{(\mathrm{p})_\eta(\mathrm{q})_\eta}{(\mathrm{r})_\eta(1)_{\eta+1}}$$

$$=\frac{\Gamma(\mathrm{r})\Gamma(\mathrm{r}-p-q)}{\Gamma(\mathrm{r}-p)\Gamma(\mathrm{r}-q)}\Bigg[\frac{2\rho(\mathrm{p})_2(\mathrm{q})_2}{(\mathrm{r}-p-q-2)_2}+\frac{[(2+4\rho)-\rho(1+t+t^2u_{\eta+1})(1+\gamma)]pq}{(\mathrm{r}-p-q-1)}$$
$$+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\Bigg]-[(1-\gamma)-tu_{\eta+2}(1+\gamma)],$$

which is bounded above by (1–ρ) if and only if (17) holds, which completes proof of (i).

To prove (ii) we apply lemma 2.2 to

$$\mathrm{G(p,q;r;z)}=\mathrm{z}-\frac{|pq|}{r}\sum_{\eta=2}^{\infty}\frac{(\mathrm{p}+1)_{\eta-2}(\mathrm{q}+1)_{\eta-2}}{(\mathrm{r}+1)_{\eta-2}(1)_\eta}\mathrm{z}^\eta.$$

It is suffices to show that

$$\sum_{\eta=2}^{\infty}\Big[2\rho\eta(\eta-1)(\eta-2)+[(2+4\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)]\eta(\eta-1)$$
$$+[(1-\gamma)-tu_\eta(1+\gamma)](\eta-1)+[(1-\gamma)-tu_\eta(1+\gamma)]\Big]\frac{(\mathrm{p}+1)_{\eta-2}(\mathrm{q}+1)_{\eta-2}}{(\mathrm{r}+1)_{\eta-2}(1)_\eta}\le\frac{[(1-\gamma)-tu_\eta(1+\gamma)]\mathrm{r}}{|pq|}.$$

Now

$$\sum_{k=0}^{\infty}\Big[2\rho\eta(\eta+1)(\eta+2)+[(2+4\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)](\eta+2)(\eta+1)$$
$$+[(1-\gamma)-tu_{\eta+2}(1+\gamma)](\eta+1)+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\Big]\frac{(\mathrm{p}+1)_\eta(\mathrm{q}+1)_\eta}{(\mathrm{r}+1)_\eta(1)_{\eta+2}}$$
$$=2\rho\sum_{\eta=1}^{\infty}\frac{(\mathrm{p}+1)_\eta(\mathrm{q}+1)_\eta}{(\mathrm{r}+1)_\eta(1)_{\eta-1}}+[(2+4\rho)-\rho(1+t+t^2u_{\eta+2})(1+\gamma)]\sum_{\eta=0}^{\infty}\frac{(\mathrm{p}+1)_\eta(\mathrm{q}+1)_\eta}{(\mathrm{r}+1)_\eta(1)_\eta}$$
$$+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\sum_{\eta=0}^{\infty}\frac{(\mathrm{p}+1)_{\eta+1}(\mathrm{q}+1)_{\eta+1}}{(\mathrm{r}+1)_{\eta+1}(1)_{\eta+1}}$$

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$$=\frac{\Gamma(r+1)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}\left[\frac{2\rho(p+1)(q+1)}{(r-p-q-1)(r-p-q-2)}+\frac{[(2+4\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)]}{r-p-q-1}+\frac{[(1-\gamma)-tu_{\eta+2}(1+\gamma)}{pq}\right]-\frac{[(1-\gamma)-tu_{\eta+2}(1+\gamma)]r}{pq}$$

$$=\frac{\Gamma(r+1)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}\left[\frac{2\rho(p+1)(q+1)}{(r-p-q-1)(r-p-q-2)}+\frac{[(2+4\rho)-\rho(1+t+t^2u_\eta)(1+\gamma)]}{r-p-q-1}+\frac{[(1-\gamma)-tu_{\eta+2}(1+\gamma)}{pq}\right]$$
$$\le\frac{[(1-\gamma)-t(1+\gamma)]r}{|pq|}+\frac{[(1-\gamma)-tu_{\eta+2}(1+\gamma)]r}{pq}.$$

which completes the proof.

For case of ρ=0, we get the following.

Corrollry 3.1.

(i) If p, q > 1 and r > p+q+1, a sufficient condition for G(p, q; r; z) defined by (17) which stated to be the Subclass C(γ,t) is that

$$\frac{\Gamma(r)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}\left[\frac{2pq}{(r-p-q-1)}+(1-\gamma)-tu_{\eta+2}(1+\gamma)\right]$$
$$\le[(1-\gamma)-t(1+\gamma)]+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]. \tag{20}$$

(ii) If p, q > –1, pq < 0 and r > max{0, p+q+1}, if G(p, q; r; z) defined by (16) which stated to be the Subclass C_T(ρ,γ,t) if and only if

$$\frac{\Gamma(r+1)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}\left[\frac{2}{(r-p-q-1)}+\frac{(1-\gamma)-tu_{\eta+2}(1+\gamma)]}{pq}\right]$$
$$\le[(1-\gamma)-t(1+\gamma)]\frac{r}{|pq|}+[(1-\gamma)-tu_{\eta+2}(1+\gamma)]\frac{r}{pq}.$$

IV. FINDINGS

We introduce new subclasses of Sakaguchi type function in the class B(ρ,γ,t). In the present paper, we determine sufficient conditions for zF(p, q; r; z) to be in B(ρ,γ,t) and also give necessary and sufficient conditions for zF(p, q; r; z) to be in B(ρ,γ,t) with appropriate restrictions on p, q, r. Furthermore, we consider an integral operator related to the hypergeometric function.

V. CONCLUSION

A modest attempt has been made in this report to study certain class of analytic functions and Sakaguchi type functions on the open unit disk . Introduced new Subclasses of analytic univalent functions in the class B(ρ,γ,t) and by selecting the values(γ ∈ R, 0 ≤ ρ ≤ 1 and |t|≤1, t≠1), with negative coefficient and obtained conditions, integral operator using Hypergeometric functions.

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REFERENCES

1. R. Bharathi, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and a corresponding class of starlike functions, *Tamkang J. Math.*, **28** (1997), 17–32.
2. B.C. Carlson and D.B. Shaffer, Starlike and prestarlike hypergeometric functions, *J. Math. Anal. Appl.*, **15** (1984), 737–745.
3. N.E. Cho, S.Y. Woo and S. Owa, Uniform convexity properties for hypergeometric functions, *Frac. Cal. Appl. Anal.*, **5(3)** (2002), 303–313.
4. A.W. Goodman, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), 87–92.
5. A.W. Goodman, On uniformly starlike functions, *J. Math. Anal. Appl.*, **155** (1991), 364–370.
6. W. Ma and D. Minda, On uniformly convex functions, *Ann. Polon. Math.*, **57** (1992), 166–175.
7. E. Merkes and B.T. Scott, Starlike hypergeometric functions, *Proc. Amer. Math. Soc.*, **12** (1961), 885–888.
8. F. Rønning, On starlike functions associated with parabolic regions, *Ann. Univ. Marie Curie-Sklodowska Sect.*, **A 45** (1991), 117–122.
9. F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118** (1993), 190–196.
10. S. Ruscheweyh and V. Singh, On the order of starlikeness of hypergeometric functions, *J. Math. Anal. Appl.*, **113** (1986), 1–11.
11. H. Silverman, Starlike and Convexity properties for hypergeometric functions, *J. Math. Anal. Appl.*, **172(3)** (1993), 574–581.
12. H.M. Srivastava and A.K. Mishra, Applications of fractional calculus to parabolic starlike and uniformly convex functions, *Comput. Math. Appl.*, **39** (2000), 57–69.
13. B.Srutha Keerthi , B.Adolf Stephen, A.Ganghadaran and S.Sivasubramanian, Some Properties of Hypergeometric functions for certain Subclasses of Analytic Functions, *Indian J. Math.,* **50(3)** (2008), (495-504).

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