Coefficient Inequlity for New Subclass of Sakaguchi Type Function Related To Sigmoid Functions

P. Mini, B. Srutha Keerthi

***Abstract:** The object of the present paper is to obtain initial coefficients $|a_2|,|a_3|,|a_4|$, upper bounds of $|a_3 - \mu a_2^2|$ and second Hankel determinant associated with a class of analytic univalent function of sakaguchi type function related to sigmoid function in the open unit disc Δ. Various authors as Abiodum, Tinuoye Oladipo, Murugusundaramoorthy et. al., and Olatunji have studied sigmoid function for different classes of analytic and univalent functions. Our results serves as a generalisation in this direction and it gives birth some existing subclasses of functions.*

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***Keywords and Phrases:** Analytic function, Starlike function, Convex function, univalent function, coefficient estimate, subordination, upper bound, sigmoid function, differential operator, second Hankel determinant.*

I. INTRODUCTION

The theory of special function has over shining by other fields like real analysis, functional analysis, topology, algebra, differential equations and so on. The generalized hypergeometric functions plays a major role in geometric function theory after the proof of Bieberbach Conjecture by de-Branges. Though the special functions does not have a specific definition, its application widely extend to physics, computer etc. There are various special functions but we shall concern with one of the activation function known as sigmoid function or simple logistic function. It is more popular because of its gradient descendent learning algorithm. Sigmoid function is the most commonly known function used in feed forward neural networks because of its non-linearity and the compuatational simplicity of its derivative. Activation function is an in information process consisting of a large number of interrelated processing elements(neurons), inspired by the same way biological nervous system(such as brain), working together to solve a specific task. The function can be learned by example, but cannot be programmed to do specific tasks. It can be evaluated in different ways, most specially by truncated series expansion. This function can be categorised into three, namely, ramp function, threshold function and sigmoid function. The sigmoid function of the form

$$h(z) = \frac{1}{1+e^{-z}} \tag{1.1}$$

is differentiable and has the following properties:

- It outputs real numbers between 0 and 1.
- It maps a very large input domain to a small range of outputs.
- It never loses information because it is an injective function.
- It increases monotonically.

Sigmoid function is perfectly useful in geometric function theory with all the four properties.

Let A be the class of all univalent analytic functions f of the for

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.2}$$

defined in the open unit disk $\Delta = \{z : z \in C \text{ and } |z| < 1\}$, and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ for $f \in A$. Recall that S^* and C denotes the class of star like and convex functions which their geometric condition satisfies

$Re\left\{1 + \frac{zf'(z)}{f(z)}\right\} > 0$ and $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$, $z \in \Delta$.

Several authors have used the above two classes of functions in different ways of perspectives.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, if there is an analytic function $w : \Delta \to \Delta$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. It follows from Schwarz lemma that $f(z) \prec g(z)$ $(z \in \Delta) \Rightarrow f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. If $g(z)$ is univalent, then $f(z) \prec g(z)$, $(z \in \Delta)$ $\leftrightarrow$ $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Let P be the class of Caratheodory function with positive real part consisting of all analytic functions $p : \Delta \to C$ satisfying $p(0) = 1$ and $Re\ p(z) > 0$. We need the following results about the functions belonging to the class P.

If the function $p \in P$ is given by the series

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + . . . , \tag{1.3}$$

then the following sharp estimates holds;

$$|c_n| \leq 2, \quad (n = 1,2,3, . . .). \tag{1.4}$$

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In 1976, Noonan and Tomas [5] stated the q[th] Hankel determinant for q ≥1 and n ≥1 are defined by

$$H_q(n) = \begin{bmatrix} a_n & a_{n+1} & . & . & a_{n+q-1} \\ a_{n+1} & a_{n+2} & . & . & a_{n+q} \\ . & . & . & . & . \\ a_{n+q-1} & a_{n+q} & . & . & a_{n+2q-2} \end{bmatrix}$$

In recent years , several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent functions . The Hankel determinant $H_2(1) = a_3 - a_2^2$ is the well known Fekete-Szegö functional. For results related to this functional, see. The second Hankel determinant $H_2(2)$) is given by $H_2(2) = a_2a_4 - a_3^2$.

For the purpose of our results, the following lemma shall be necessary.

Lemma 1.1 . *Let h be a sigmoid function and*

$$\phi(z) = 2h(z) = 1+\sum_{m=1}^{\infty}\frac{(-1)^n}{z^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n\right)^m \qquad (1.5)$$

then $\phi(z) \in P$, $|z| <1$ *where* $\phi(z)$ *is modified sigmoid function*

Lemma 1.2 . *Let h be a sigmoid function and*

$$\phi_{m,n}(z) = 2h(z) = 1+\sum_{m=1}^{\infty}\frac{(-1)^n}{z^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n\right)^m \qquad (1.6)$$

then $|\phi_{m,n}(z)| < 2.$

Lemma 1.3 . *If* $\phi(z) \in P$ *and it is starlike, then f is normalised univalent function of the form (1.2), taking m=1, Fadipe et. al. remarked that*

$$\phi(z) = 1+\sum_{n=1}^{\infty}c_n z^n$$

where $c_n = \frac{(-1)^{n+1}}{2n!}$ *then* $|c_n| \le 2, (n = 1,2,3, . . .)$, *the result is sharp for each n.*

Definition 1.4.

Let the class $R^{\lambda,n,k}(b,\phi_{m,n}), b \in \setminus C \{0\}$ denote the subclass of A consisting of functions f of the form (1.2) satisfying

$$Re\left[1+\frac{1}{b}\left[\frac{[(1-t)z]^{1-\lambda}\,(D_{n-1}f'(z))}{[(D_{n-1}f(z))-t(D_{n-1}f(tz))]^{1-\lambda}}-1\right]\right] > \alpha, \qquad (1.7)$$

for $0 \le \lambda \le 1$, $|t| \le 1, t \ne 1$ and $\phi_{m,n}(z)$ is a simple logistic sigmoid activation function [9] and

$$D_{n-1}f(z) = \frac{z}{(1-z)^{n+1}} * f(z) \qquad (1.8)$$

when $n+1 > 0$.

II. COEFFICIENT ESTIMATES

Various authors as Abiodum [6], Tinuoye Oladipo [3], Murugusundaramoorthy et. al.[4], and Olatunji [7,8] have studied sigmoid function for different classes of analytic and univalent functions. In this paper, we obtain few coefficient bounds for the class $R^{\lambda,n,k}(b,\phi_{m,n})$,

Theorem 2.1. If $f(z)$ given by (1.2) belongs to the class $R^{\lambda,n,k}(b,\phi_{m,n}), m \ge 2$ then,

$$|a_2| \le \frac{|b|}{2(n+1)[2-(1-\lambda)(1+t)]}, \qquad (2.1)$$

$$|a_3| \le \frac{(1-\lambda)(1+t)|b|^2[4-(2-\lambda)(1+t)]}{4(n+1)(n+2)[3-(1-\lambda)(1+t+t^2)][2-(1-\lambda)(1+t)]^2}, \qquad (2.2)$$

$$|a_4| \le \frac{|b|}{(n+1)(n+2)(n+3)[4-(1-\lambda)(1+t)(1+t^2)]}\times$$

$$\left[\frac{-(1-\lambda)(1+t)b^2\left\{\begin{matrix}18\lambda(1+t)-6\lambda(2-\lambda)(1+t)^2\\ +(1-\lambda)(2-\lambda)(1+t)(1+t+t^2)*\\ [(1+t)(2\lambda-3)+12]-24(1-\lambda)(1+t+t^2)\end{matrix}\right\}}{8[3-(1-\lambda)(1+t+t^2)][2-(1-\lambda)(1+t)]^3}-\frac{1}{4}\right]. \qquad (2.3)$$

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Proof. If $f(z) \in R^{\lambda,n,k}(b,\phi_{m,n})$,, then

$$1+\frac{1}{b}\left[\frac{[(1-t)z]^{1-\lambda}(D_{n-1}f^{'}(z))}{[(D_{n-1}f(z))-t(D_{n-1}f(tz))]^{1-\lambda}}-1\right]=\phi_{m,n}(z), \qquad (2.4)$$

where Taylor's series expansion of $\phi_{m,n}(z)$ gives

$$\phi_{m,n}(z)=1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{17}{40320}z^7+\ldots, \qquad (2.5)$$

from (2.4) we have

$$\begin{aligned}1+\frac{1}{b}\Bigg\{&1-[(n+1)(1-\lambda)(1+t)a_2+2a_2(n+1)]z\\
&+\left[\begin{array}{c}\frac{(1-\lambda)(2-\lambda)(1+t)^2(n+1)^2a_2^2}{2}-\frac{(1-\lambda)(1+t+t^2)(n+1)(n+2)a_3}{2}\\ -2(1-\lambda)(1+t)(n+1)^2a_2^2+\frac{3a_3(n+1)(n+2)}{2}\end{array}\right]z^2\\
&+\Bigg[\frac{-(1-\lambda)(n+1)(n+2)(n+3)(1+t)(1+t^2)a_4}{6}\\
&+\frac{(1-\lambda)(2-\lambda)(1+t)(1+t+t^2)(n+1)^2(n+2)a_2a_3}{2}-\frac{(1-\lambda)(2-\lambda)(3-\lambda)(1+t)^3(n+1)^3a_2^3}{6}\\
&-(1-\lambda)(1+t+t^2)(n+1)^2(n+2)a_2a_3+\frac{2(n+1)^3(1-\lambda)(2-\lambda)(1+t)^2a_2^3}{2}\\
&-\frac{3(1-\lambda)(1+t)(n+1)^2(n+2)a_2a_3}{2}+\frac{2a_4(n+1)(n+2)(n+3)}{2}\Bigg]z^3+\ldots\Bigg\}=\phi_{m,n}(z).\end{aligned}$$

Equating the coefficients of $z,\ z^2$ and z^3, we obtain

$$|a_2|\le\frac{|b|}{2(n+1)[2-(1-\lambda)(1+t)]}, \qquad (2.6)$$

$$|a_3|\le\frac{(1-\lambda)(1+t)|b|^2[4-(2-\lambda)(1+t)]}{4(n+1)(n+2)[3-(1-\lambda)(1+t+t^2)][2-(1-\lambda)(1+t)]^2}, \qquad (2.7)$$

$$|a_4|\le\frac{|b|}{(n+1)(n+2)(n+3)[4-(1-\lambda)(1+t)(1+t^2)]}\times$$

$$\left[\frac{-(1-\lambda)(1+t)b^2\left\{\begin{array}{c}18\lambda(1+t)-6\lambda(2-\lambda)(1+t)^2\\ +(1-\lambda)(2-\lambda)(1+t)(1+t+t^2)*\\ [(1+t)(2\lambda-3)+12]-24(1-\lambda)(1+t+t^2)\end{array}\right\}}{8[3-(1-\lambda)(1+t+t^2)][2-(1-\lambda)(1+t)]^3}-\frac{1}{4}\right]. \qquad (2.8)$$

Results (2.1),(2,2) and (2.3) can be obtained from (2.6), (2.7) and (2.8) respectively.

Corollary 2.2. *If* $f(z)\in T(b,\phi)$, *then [10]*

$$|a_2|\le\frac{|b|}{2(n+1)}, \qquad (2.9)$$

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$$|a_3| \leq \frac{|b|^2}{4(n+1)(n+2)} \quad , \tag{2.10}$$

$$|a_4| \leq \frac{|b|}{3(n+1)(n+2)(n+3)}\left[\frac{3b^2}{8}-\frac{1}{4}\right]. \tag{2.11}$$

III. FEKETE-SZEGÖ INEQUALITIES

Recently there has been interest to obtain the Fekete-Szegö inequality for various subclasses of S and C. In this section making use of a_2 and a_3, we prove the following Fekete-Szegö result for the function class $R^{\lambda,n,k}(b,\phi_{m,n})$,

Theorem 3.1. *If $f(z)$ belongs to the class $R^{\lambda,n,k}(b,\phi_{m,n})$, of the form (1.2), then*

$$|a_3-\mu a_2^2| \leq \frac{|b|^2}{4(n+1)^2(2-(1-\lambda)(1+t))^2} * \left[\frac{(1-\lambda)(1+t)(n+1)[4-(2-\lambda)(1+t)]}{(n+2)[3-(1-\lambda)(1+t+t^2)]}-\mu\right]. \tag{3.1}$$

Proof. From (2.6) and (2.7)

$$|a_3-\mu a_2^2| = \frac{|b|^2}{4(n+1)^2(2-(1-\lambda)(1+t))^2} * \left[\frac{(1-\lambda)(1+t)(n+1)[4-(2-\lambda)(1+t)]}{(n+2)[3-(1-\lambda)(1+t+t^2)]}-\mu\right] , \tag{3.2}$$

hence (3.1) can be easily obtained from (3.2).

Corollary 3.2. *If $f(z) \in T(b,\phi)$, then [10]*

$$|a_3-\mu a_2^2| \leq \frac{|b|^2}{4(n+1)^2}\left[\frac{(n+1)}{(n+2)}-\mu\right] .$$

IV. SECOND HANKEL DETERMINANT

In this section making use of a_2 and a_3, we obtain the following second Hankel determinant result for the function class $R^{\lambda,n,k}(b,\phi_{m,n})$,

Theorem 4.1 If $f(z) \in R^{\lambda,n,k}(b,\phi_{m,n})$, then

$$|a_2a_4-a_3^2| \leq \left\| \begin{matrix} \dfrac{b^2}{16(n+1)^2(n+2)(n+3)[2-(1-\lambda)(1+t)]^4[4-(1-\lambda)(1+t)(1+t^2)]} * \\ \left[\begin{matrix} -2[2-(1-\lambda)(1+t)]^3-\dfrac{b^2(1+t)(1-\lambda)}{[3-(1-\lambda)(1+t+t^2)]} * \\ \left\{\begin{matrix} 18\lambda(1+t)-6\lambda(2-\lambda)(1+t)^2+(1-\lambda)(2-\lambda)(1+t) * \\ (1+t+t^2)[((1+t)(2\lambda-3)+12)-24(1-\lambda)(1+t+t^2)] \\ +\dfrac{b^2(1-\lambda)^2(1+t)^2(n+3)[4-(2-\lambda)(1+t)]^2[4-(1-\lambda)(1+t)(1+t^2)]}{(n+2)[3-(1-\lambda)(1+t+t^2)]} \end{matrix}\right\} \end{matrix}\right] \end{matrix} \right\| . \tag{4.1}$$

Proof. From (2.6) , (2.7) and (2.8), we have

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$$a_2a_4 - a_3^2 = \left\| \begin{array}{c} \dfrac{b^2}{16(n+1)^2(n+2)(n+3)[2-(1-\lambda)(1+t)]^4[4-(1-\lambda)(1+t)(1+t^2)]} * \\ \left[\begin{array}{c} -2[2-(1-\lambda)(1+t)]^3 - \dfrac{b^2(1+t)(1-\lambda)}{[3-(1-\lambda)(1+t+t^2)]} * \\ \left\{ \begin{array}{c} 18\lambda(1+t) - 6\lambda(2-\lambda)(1+t)^2 + (1-\lambda)(2-\lambda)(1+t) * \\ (1+t+t^2)[((1+t)(2\lambda-3)+12) - 24(1-\lambda)(1+t+t^2)] \\ + \dfrac{b^2(1-\lambda)^2(1+t)^2(n+3)[4-(2-\lambda)(1+t)]^2[4-(1-\lambda)(1+t)(1+t^2)]}{(n+2)[3-(1-\lambda)(1+t+t^2)]} \end{array} \right\} \end{array} \right] \end{array} \right\|, \quad (4.2)$$

which gives the desired inequality (4.1)

□

Corollary 4.2. *If* $f(z) \in T(b,\phi)$, *then [10]*

$$|a_2a_4 - a_3^2| \le \left| \frac{b^2}{24(n+1)^2(n+2)(n+3)} \left[-1 + \frac{3b^2}{2} - \frac{3b^2(n+3)}{2(n+2)} \right] \right|. \quad (4.3)$$

□

V. FINDINGS

We introduced new subclasses of analytic univalent Sakaguchi type function. In the present paper is to obtain initial coefficients $|a_2|, |a_3|, |a_4|$, upper bounds of $|a_3 - \mu a_2^2|$ and second Hankel determinant associated with a class of analytic univalent function of sakaguchi type function related to sigmoid function in the open unit disc Δ.

VI. CONCLUSION

By selecting the values of ρ and t we state the interesting Fekete-Szegö inequality and Second Hankel determinant for the subclasses of $C\,(\tau\,;\,\varphi)$ [1,2].The results above serve as a new generalisation of subclasses of univalent functions related to sigmoid functions. The investigation of initial coefficients bounds, Fekete-Szegö inequality and Second Hankel determinant for various subclasses can be a scope of future research.

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