Renormalization in AC Circuits based on Fractal

R. Kamali, G. Jayalalitha

Abstract--- ***In this paper the Feynman infinite ladder AC circuits is analyzed by Renormalization method to form Fractal AC circuits, since operative of Renormalization gives the potentialities and interrelations of an infinite ladder. In particular, this analyzation is for the so-called Feynman Sierpinski ladder that exhibits the AC frequency response of Sierpinski Gasket networks. This extends the self-similarity resistance networks. There forms a Regular Set which is rectifiable and Line Graphs are also formed using adjacent edges of the AC circuits induces the connectedness and the continuous self-similarity throughout the circuit.***

Keywords--- ***Fractals, AC circuits, Renormalization, Regular Set, Line Graphs, Rectifiable, Iteration.***

I. INTRODUCTION

Application of linear networks mainly electrical circuits have been considered in many research fields such as Physics, Engineering also in Mathematics. For an infinite network, Feynman constructed a new infinite ladder circuit as described in Figure-1 whose behavior is surprising [1]. This infinite ladder circuit is made of purely imaginary impedances of capacitors and impedances of inductors with a connection allowing the free flow of electrons throughout the network [2]. This theory extends the analysis of self-similar among the resistance networks which was introduced by Fukushima, Kigami, Kusuoka and more recently studied by Strichartz et.al [3, 4]. It also extends the concepts of Fractal folds and Fractal quantum graph [5] to the alternating current networks. The term Fractal was coined by Benoit Mandelbrot in 1975 and was derived from the Latin fractus meaning *broken* or *fractured*, that is it's a geometric shape which can be split into parts, each of which is a reduced- size copy of the whole this is called self–similarity which is the main character in Fractals. Few examples of Fractals are Cantor set, Vonkoch Curve, Sierpinski Gasket. In this Sierpinski Gasket is the simplest form of a Fractal network that supports an infinite Dirichlet forms.

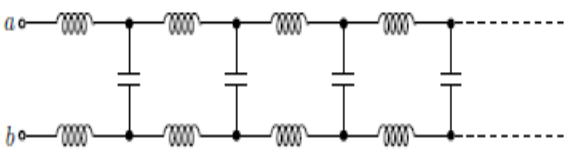

Figure 1: Feynman's infinite ladder Circuit

The formation of non-trivial resistance forms on the Sierpinski Gasket, as it is operationally an aggregation process obtained by a simple iterative process which is obtained as a resistor in graphs estimated to the Fractal [6]. In addition for any two-sided AC circuit on the Sierpinski Gasket there exists numerous number of self-similar circuits within it by making infinite number of iterations [7, 8, 9]. This creation of self-similarity circuits in the Serpinski Gasket forms a Fractal structure in it.

Our work in this paper is to analyze this infinitely-iterated Fractal Sierpinski Gasket by Renormalizing the AC circuits at different scales. It is possible to renormalize the AC circuits, as Renormalization [10, 11] is nothing but a simple collection of systems in significant field theory of self-similar geometric structures which gives infinite quantities by varying values to recompense the properties of their self-interactions. The fact that they are self-similar makes them multi-fractals in the sense it propounds an idea that the exact Renormalization can be exists in different scales. This Renormalization procedure is based on the requirement that certain physical quantities are equal to the observed values. Also this AC circuits is rectifiable which forms a regular set [12] at different scales. This formation of regular set makes renormalization simpler to the entire AC circuit. Fractal graphs as obtained forms a Line graph, with its vertices as the edges and any two vertices of it are adjacent, if the corresponding edges are incident [13, 14] which shows that the graph is a connected graph. This connection in the circuit leads to a continuous construction of the AC circuits which give a spectral analysis.

In section 2, the Feynman-Sierpinski Ladder circuit is explained in detail. In section 3 and section 4, the physical quantities of AC circuits are analyzed by Fractal Renormalization and the Regularity property and Line Graphs for the infinite ladder is discussed with a suitable example.

II. FEYNMAN-SIERPINSKI LADDER CIRCUIT

As Fractals are infinitely complex patterns that are self-similar across different scales and by associating with the Feynman ladder circuit a Feynman-Sierpinski Ladder circuit is constructed by the following substitution procedure as shown in Figure-2.

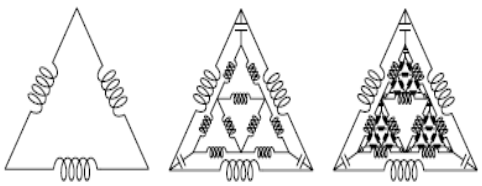

Figure 2: Construction of Feynman-Sierpinski Ladder circuit

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This ladder is exposed in a triangular form by joining the mid-value of the boundary line of the triangle and within the triangle it is connected using capacitors of impedance $Z_C = \frac{1}{i\,\omega C}$ and inductors of impedance $Z_L = i\omega L$ as indicated in the central image of Figure-2. This process of iteration is repeated for infinite number of times. As a result a set of self-similarity occurs in the Feynman-Sierpinski Ladder circuit with the same values of capacitors and inductors throughout the circuit.

Let $\frac{2}{3}Z$ denote the characteristic impedance across the two external vertices in the limiting structure. This value is chosen due to the reason that it would be the effective impedance across each edge of the initial triangle. This shows that each choice of capacitance and inductance determines a unique Z that satisfies the following equation (1)

$$\frac{1}{Z} = \frac{1}{Z_L} + \frac{1}{3Z_C + {}^{5Z}/_{3}} \quad (1)$$

with the condition under which Z has positive real part which indicates that the circuit is a filter, which shows that there is a free flow of electrons in the circuit. This leads to the formation of continuous flow of electrons throughout the AC circuit.

Subsequently if the capacitances Z_C, the inductances Z_L and also if the applied AC signal has frequency, such that $Z_C = \frac{1}{i\,\omega C}$ and $Z_L = i\omega L$, then there exists a solution in which Z has positive real part exactly when

$$9\left(4-\sqrt{15}\right) < 2\omega^2 LC < 9\left(4+\sqrt{15}\right) \quad (2)$$

in this case the impedance is

$$Z = \frac{1}{10\omega C}\left((9+2\omega^2 LC)i + \sqrt{144\omega^2 LC - 4(\omega^2 LC)^2 - 81}\right) \quad (3)$$

From the equation (2), it is clear that Z splits into two parts, one is purely imaginary and the other is purely real.

When Z is purely imaginary it is given by

$$Z = \frac{i}{10\omega C}\left(2\omega^2 LC + 9 - \sqrt{4(\omega^2 LC)^2 + 81 - 144\omega^2 LC}\right) \quad (4)$$

for $2\omega^2 LC < 9\left(4-\sqrt{15}\right)$

and when Z is purely real it is given by

$$Z = \frac{i}{10\omega C}\left(2\omega^2 LC + 9 + \sqrt{4(\omega^2 LC)^2 + 81 - 144\omega^2 LC}\right) \quad (5)$$

for $2\omega^2 LC > 9\left(4+\sqrt{15}\right)$

III. FRACTAL RENORMALIZATION

The main idea of Renormalization is the occurrence of self-similarity for different scales. This similarity breaks down for large sizes as well as for small sizes of the order of the distance r between nearest neighbors on the lattice. A different modification of the Sierpinski Gasket for different scales is made in the AC circuits in which each scaling represents a circuit consisting of a resistor with inductance Z_L and a capacitor with capacitance Z_C. Here the circuit construction is formed by connecting the symmetric inverted Y-circuits, in which the upright division has impedance Z_1 and the two slanting divisions each having impedance Z_2 with each branch containing $n-1$ branches as shown in Figure-3.

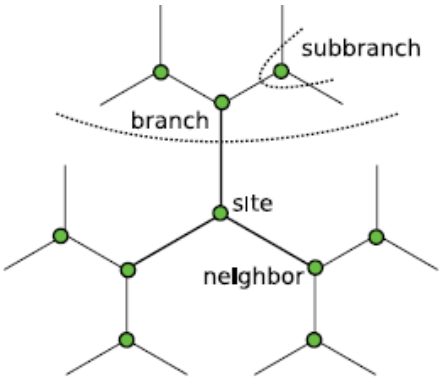

Figure 3: Fractal Y-circuit

The graph in Figure-3 is thus succeeded by rescaling the preceding one and constructing three more copies inside the Sierpinski triangle as in the central diagram of Figure-2. The impedances on each copies are rescaled by a real factor $r>0$, with the condition that the characteristic impedances of the first and the second circuits are equivalent. A Symmetrical work is made with the impedances from the lower vertex to the two upper vertices and between the two upper vertices. Then by using Kirchhoff's laws, the following system of equations relating Z_1 and Z_2 are obtained

$$Z_1 + \frac{Z_2}{2} = rZ_1 + \frac{rZ_1 + 2rZ_2 + \frac{1}{i\omega C}}{2} \quad (6)$$

and

$$2Z_2 = 2rZ_2 + \left(\frac{1}{2rZ_2 + i\omega L} + \frac{1}{2rZ_1 + 2rZ_2 + 2/(i\omega C)}\right)^{-1} \quad (7)$$

If (6) and (7) holds, then $r \neq 1$.

Also if $r \neq \frac{2}{3}$, then

$$Z = \frac{1}{2-3r}\left((2r-1)Z_2 + Z_C\right) \quad (8)$$

and

$$r(5r-3){Z_2}^2 + (2r-1)(2Z_C + Z_L)Z_2 + Z_C Z_L = 0 \quad (9)$$

while if $r = \frac{2}{3}$, then

$$Z_2 = -3Z_C \quad (10)$$

and

$$Z_1 = \frac{3Z_C(Z_L + 2Z_C)}{4(Z_L - 2Z_C)} \quad (11)$$

For an AC circuit with frequency ω, C the capacitance and L the inductance of the components used in the construction is such that $Z_C = \frac{1}{i\,\omega C}$ and $Z_L = i\omega L$ and identify that for what values of r for which the impedances measured in the circuit have positive real part. This positive real value holds the condition that the circuit is filter; this condition reveals the continuous free flow of electrons throughout the circuit.

With this assumptions for $r \geq \frac{3}{5}$ and $r \neq 1$, there are solutions for $Z_1$and Z_2, but the filter condition fails because Z_2 is purely imaginary for all ω and for $r \in \left(0, \frac{3}{5}\right)$, the circuit is a filter precisely in the frequency range

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$$\gamma(r) - \sqrt{[\gamma(r)]^2 - 1} < 2LC\omega^2 < \gamma(r) + \sqrt{[\gamma(r)]^2 - 1} \qquad (12)$$

where $\gamma(r) = 1 + \frac{2(3-5r)}{(2r-1)^2}$.

IV. REGULAR SET AND LINE GRAPHS

The following Sections 4.1 and 4.2 explains about the formation of a Regular Set in the AC circuit and the formation of a Line graph based on the partition and the adjacent properties based on the edges of the AC circuit.

4.1 Formation of Regular sets in the AC circuit

The Y circuit formed in Figure-3 is used in reducing a complicated network to a simpler one, though there are of course networks for which it is not effective. Also from Figure-3, the Y-circuit has two distinct points, one containing the point and the other containing the closed sets. This implies that there exists disjoint neighbourhoods separating one point from the other with a curve joining these points as shown in Figure-4. As rectifiable curve is a curve of finite length and let Z be a continuum with *1*-dimensional Hausdroff measure of Z less than infinity. Then *Z* consists of a countable union of rectifiable curves, along with a set of *1*-dimensional Hausdroff measure zero.

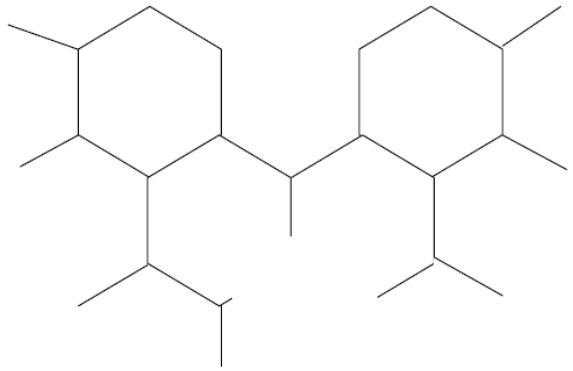

Figure 4: Fractal Regular Y-circuit with neighbouring sites

A rectifiable curve is also a regular *1*-set since *Z* contains at least two distinct points with *1*-dimensional Hausdroff measure > 0, so *Z* is a*1*-set. As *1*-set contained in a countable union of rectifiable curves is a Y-set indicates that Y-set is a Regular set. Any point in a set where both the sides are one and the same then that point is called a Regular point. Almost all the points in this Y-circuit are regular. Since this graph *Z* is a Regular graph and both sides of the bipartition have the same number of vertices with equal degrees, this graph is also strongly regular.

This shows that the entire AC circuit forms a Regular set. Due to the existence of Regular set, the infinite resistance is proved to be continuous throughout the AC circuit and this continuity in the AC circuit leads to the formation of an infinite Fractal AC circuit.

4.2 Formation of Line Graphs

Considering the impedance *Z* as a graph such that *Z = (V, E)* is a graph with vertex set *V = V (Z)* and edge set *E = E(Z)*. There forms a line graph *L(Z)* for a graph *Z* with vertices, the edges of Z and any two vertices are adjacent in *L(Z)* if the corresponding edges are adjacent in *Z*.

Let the sub graph z' induced by $\{v_1, v_2, \ldots, v_k\} \subseteq V$ is denoted by $<v_1, v_2, \ldots, v_k>$. The graph *Z* is connected since there is at least one path connecting any two of its edges which translates into a path in *L*(*Z*) containing any two of the vertices of *L*(*Z*).

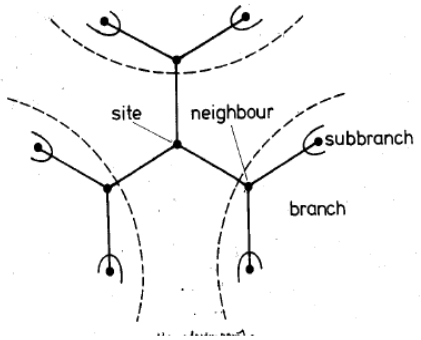

Figure 5: Impedance Graph Z

This impedance graph *Z* formed in the AC circuit as shown in Figure-5 has adjacent edges with equidistant and there is a continuous partition which is connected throughout the circuit confirms the formation of a Line graph in the AC circuit, this can be proved by the following theorem.

Theorem 4.1

For a connected graph Z with exactly 2n odd vertices, there exists n edge-disjoint subgraphs such that they together contain all edges of Z and that each is a unicursal graph.

Proof

A unicursal graph is a graph which is connected by a unicursal line.

Let us consider the odd vertices for the graph *Z* formed as in Figure-4 be $v_1, v_2, \ldots, v_n$; $w_1, w_2, \ldots, w_n$ in any arbitrary order.

In this graph *Z* just add a set of *n* edges between the vertex pairs

$(v_1, w_1), (v_2, w_2), \ldots, (v_1, w_n)$ to form a new graph z'.

As every vertex of z' is of even degree, z' consists of an Euler line ρ. Suppose from this ρ, if the newly added *n* edges are removed, then ρ would be split into *n* walks as walk is a finite alternating sequence of vertices and edges such that each edge is incident with the vertices preceding and following it and each of which is a unicursal line.

The first removal of *n* edges leaves a single unicursal line. Likewise the second time removal of *n* edges makes a split into two unicursal lines and each successive removal will split unicursal lines, until there are *n* of them.

Hence these unicursal lines are connected and form the unicursal graph

By the theorem4.1 it is clear that there forms a Line graph in the AC circuit which induces the connection throughout the circuit as in the Fig4. This is just a part of the AC circuits which can be extends to the entire infinite AC circuits.

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This connection leads to the formation of a Self- Similar continuous AC circuits because of this extension of Self-Similarity throughout the circuit there exists a Fractal structure in the AC circuit since Self-Similarity plays a major role in Fractals and encourages the continuous flow of current in the circuit.

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