# Global Synchronization in Arrays of Coupled Neural Networks with Uncertainties and Mixed Delays

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***Abstract**: this paper deal with the problem of global synchronization in arrays of coupled neural networks with uncertainties and mixed delays. By construction of a suitable Lyapunov-Krasovskii's functional (LKF), Kronecker product properties and utilization of Wirtinger's inequality, novel delay-dependent criteria for the robust synchronization of the networks are established in terms of linear matrix inequalities (LMIs) which can be easily solved by various effective optimization algorithms. A numerical example is given to illustrate the effectiveness of the proposed method.*

***Index Terms**: Synchronization, Neural Networks, Time-Varying Delay, Leakage Delay.*

## I. INTRODUCTION

In many real-world applications, large-scale neural networks show a surprising degree of synchronization. To unravel the underlying mechanics of synchronization in these neural networks, a generally linearly coupled network with time-varying delay and leakage delay is proposed [1-5]. The problem of synchronization of neural networks which is one of hot research fields of networks has been a challenging issue due to its potential applications such as information science, engineering, biological systems and so on [6-7]. Therefore, this paper aims to study the global synchronization analysis of coupled neural networks with interval time-varying delays and leakage delay. Based on delay partitioning, a LKF is constructed to obtain several improved delay conditions which guarantee the robust synchronization of coupled neural networks. We propose the networks by additional useful terms with uncertainties, the leakage delay and estimated some integral terms by Wirtinger's inequality provided a tighter lower bound than Jensen's inequality. Consider the following dynamical network consisting of $N$ coupled nodes can be described as follows:

$$\begin{cases} \dot{x}_i(t) = -\left(A+\Delta A(t)\right)x_i(t-\sigma)+\left(W_1+\Delta W_1(t)\right)f\left(x_i(t)\right)+\left(W_2+\Delta W_2(t)\right)f\left(x_i\left(t-\tau(t)\right)\right)+\sum_{j=1}^{N} g^{(1)}_{ij}\left(\Gamma_1+\Delta\Gamma_1(t)\right)x_j(t) \\ \qquad +\sum_{j=1}^{N} g^{(2)}_{ij}\left(\Gamma_2+\Delta\Gamma_2(t)\right)x_j(t-\tau(t))+\sum_{j=1}^{N} g^{(3)}_{ij}\left(\Gamma_3+\Delta\Gamma_3(t)\right)\int_{t-\tau(t)}^{t} x_j(s)ds, \qquad i=1,2,\dots,N, \end{cases}$$

(1)

where $x_i(t)=\left[x_{i1}(t),\ \dots\ ,x_{in}(t)\right]\in\mathbb{R}^n$ is the state of the neural. $A=\text{diag}\left(a_1,\dots,a_n\right)>0$ is a diagonal matrix. $W_1$ and $W_2$ are connection weight matrices. $f(\cdot)=\left(f_1(\cdot),\dots,f_n(\cdot)\right)^T$ are the activation functions with $f(0)=0$. $\Gamma_1,\Gamma_2,\Gamma_3\in\mathbb{R}^{n\times n}$ are the inner-coupling matrix. $\tau(t)$ and $\sigma$ are the interval time-varying delays and leakage delay and satisfy $0<\tau_1\le\tau(t)\le\tau_2,\ -\infty<\mu_1\le\dot{\tau}(t)\le\mu_2,\ 0<\sigma,\ \forall t\ge 0,$ where $\tau_1,\tau_2,\mu_1,\mu_2$ and $\sigma$ are known scalars. $\Delta A(t),\Delta W_1(t),\Delta W_2(t),\Delta\Gamma_1(t),\Delta\Gamma_2(t)$ and $\Delta\Gamma_3(t)$ are parameter uncertainties. $G^{(r)}=\left(g_{ij}^{(r)}\right)_{N\times N}$, $r=1,2,3$ are the outer-coupling matrix and satisfy $g_{ij}^{(r)}=g_{ij}^{(r)}\ge 0\,(i\ne j),\ g_{ii}^{(r)}=-\sum_{j=1,j\ne i}^{N} g_{ij}^{(r)}, i,j=1,2,\dots,N.$

Throughout, we make the following assumption:

(H1) : The time-varying uncertainties $\Delta A(t),\Delta W_1(t),\Delta W_2(t),\Delta\Gamma_1(t),\Delta\Gamma_2(t)$ and $\Delta\Gamma_3(t)$ are of the form:

$$\left[\Delta A(t)\ \Delta W_1(t)\ \Delta W_2(t)\ \Delta\Gamma_1(t)\ \Delta\Gamma_2(t)\ \Delta\Gamma_3(t)\right]=HF(t)\left[E_A\ E_{w_1}\ E_{w_2}\ E_{\Gamma_1}\ E_{\Gamma_2}\ E_{\Gamma_3}\right], \tag{2}$$

where $H,E_A,E_{W_1},E_{W_2},E_{\Gamma_1},E_{\Gamma_2}$ and $E_{\Gamma_3}$ are known real constant matrices, and $F(\cdot)$ is and unknown time-varying matrix function satisfying $F^T(t)F(t)\le I$ .

(H2) **:** The neural activation functions $f_k(\cdot),\ k=1,2,\dots,n,$ satisfy $f_k(0)=0$ and for $s_1,s_2\in\mathbb{R}$, $s_1\ne s_2$,

$$l_k^-\le\frac{f_k(s_1)-f_k(s_2)}{s_1-s_2}\le l_k^+, \tag{3}$$

where $l_k^-,l_k^+$ are known real scalars.

**Definition 1** [2]: System (1) is said to be asymptotically synchronized if the following condition holds:

$$\lim_{t\to\infty}\|\ x_i(t)-x_j(t)\ \|=0,\qquad \forall\ i,j=1,2,\dots,N.$$

Let us define $x(t)=(x_1^T(t),\dots,x_N^T(t))^T$ and $f\left(x(t)\right)=\left(f^T(x_1(t)),\dots,f^T(x_N(t))\right)^T$ . Then, with the Kronecker product, we can reformulate the system (1) as follows:

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$$\begin{cases} \dot{x}(t) = -\left(I_N \otimes (A+\Delta A(t))\right)x(t-\sigma) + \left(I_N \otimes (W_1+\Delta W_1(t))\right) f(x(t)) + \left(I_N \otimes (W_2+\Delta W_2(t))\right) f\left(x(t-\tau(t))\right) \\ \qquad + \left(G^{(1)} \otimes (\Gamma_1+\Delta\Gamma_1(t))\right)x(t) + \left(G^{(2)} \otimes (\Gamma_2+\Delta\Gamma_2(t))\right)x(t-\tau(t)) + \left(G^{(3)} \otimes (\Gamma_3+\Delta\Gamma_3(t))\right)\int_{t-\tau(t)}^{t} x(s)ds. \end{cases} \tag{4}$$

**Lemma 2** [3]: For a given matrix $R>0$, the following inequality holds for any continuously differentiable function $\omega:[a,b]\to \mathbb{R}^n$

$$-\int_a^b \dot{\omega}^T(s) R\,\dot{\omega}(s)\,ds \le \frac{1}{b-a}\left(\Xi_1^T R \Xi_1 + \Xi_2^T R \Xi_2\right),$$

where $\Xi_1 = \omega(b)-\omega(a)$, $\Xi_2 = \omega(b)-\omega(a)-\frac{1}{b-a}\int_a^b \omega(s)\,ds.$

**Lemma 3** [4]: Let $\tau(t)$ be a continuous function satisfying $0\le\tau_1\le\tau(t)\le\tau_2$. For any $n\times n$ real matrix $R_1>0$ and a vector $\dot{x}:[-\tau_2,0]\to\mathbb{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $2n\times 2n$ real matrices $D$ satisfying $\begin{bmatrix} \bar{R}_1 & D \\ * & \bar{R}_1 \end{bmatrix}\ge 0,$ and

$$-(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s) R_1 \dot{x}(s)\,ds \le 2\varphi_{11}^T D\varphi_{21} - \varphi_{11}^T \tilde{R}_1 \varphi_{11} - \varphi_{21}^T \tilde{R}_1 \varphi_{21}^T,$$

where $\bar{R}_1 = \operatorname{diag}\{R_1, 3R_1\}$ and $\varphi_{11} = \begin{bmatrix} x(t-\tau(t)) - x(t-\tau_2) \\ x(t-\tau(t)) + x(t-\tau_2) - \frac{2}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} x(s)ds \end{bmatrix}.$

**Lemma 4** [4]: Let $\tau(t)$ be a continuous function satisfying $0\le\tau_1\le\tau(t)\le\tau_2$. For any $n\times n$ real matrix $R_2>0$ and a vector $\dot{x}:[-\tau_2,0]\to\mathbb{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $\phi_{i1}\in\mathbb{R}^q$ and real matrices $Z_i\in\mathbb{R}^{q\times q}, B_i\in\mathbb{R}^{q\times n}$ satisfying $\begin{bmatrix} Z_i & B_i \\ * & R_2 \end{bmatrix}\ge 0,\ i=1,2$ and

$$-\int_{t-\tau_2}^{t-\tau_1}(\tau_2-t+s)\dot{x}^T(s)R_2\dot{x}(s)ds \le \frac{1}{2}(\tau_2-\tau(t))^2\phi_{11}^T Z_1\phi_{11} + 2(\tau_2-\tau(t))\phi_{12}^T B_1\phi_{12} + \frac{1}{2}\left[(\tau_2-\tau_1)^2-(\tau_2-\tau(t))^2\right]\phi_{21}^T Z_2\phi_{21}$$
$$+2\phi_{21}^T B_2\left[(\tau_2-\tau(t))\phi_{22} + (\tau(t)-\tau_1)\phi_{23}\right],$$

where $\phi_{12} = x(t-\tau(t)) - \frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} x(s)ds,\quad \phi_{22} = x(t-\tau_1)-x(t-\tau(t)),\quad \phi_{23} = x(t-\tau_1) - \frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1} x(s)ds.$

**Lemma 5** [5]: Let $H$, $E$ and $F(t)$ be real matrices of appropriate dimensions with $F(t)$ satisfying $F(t)^T F(t)\le I$. Then, for any scalar $\varepsilon>0$:

$$HF(t)E + \left(HF(t)E\right)^T \le \varepsilon^{-1}HH^T + \varepsilon E^T E.$$

## II. LITERATURE REVIEW

Firstly, for the sake of simplicity on matrix representation $e_k = [0_{n\times(k-1)n}\ \ I_n\ \ 0_{n\times(15-k)n}],\ k=1,2,\ldots,15$ are defined as block entry matrices. The notations of several matrices are defined as:

$$\zeta(t) = \operatorname{col}\Big\{x(t),\ f(x(t)), x(t-\tau(t)), f\left(x(t-\tau(t))\right), x(t-\tau_1), f\left(x(t-\tau_1)\right), x(t-\tau_2), f\left(x(t-\tau_2)\right), \tfrac{1}{\tau_1}\int_{t-\tau_1}^{t} x(s)\,ds,$$
$$\tfrac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1} x(s)\,ds, \tfrac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} x(s)\,ds,\ \dot{x}(t-\tau_1),\ \ \dot{x}(t),\ \ x(t-\sigma),\ \tfrac{1}{\sigma}\int_{t-\sigma}^{t} x(s)\,ds\Big\},$$

$z_{ij}(t) = x_i(t)-x_j(t)$, $f(z_{ij}(t)) = f(x_i(t)) - f(x_j(t))$,

$$\zeta_{ij}(t) = \operatorname{col}\Big\{ z_{ij}(t),\ f(z_{ij}(t)),\ z_{ij}(t-\tau(t)),\ f\left(z_{ij}(t-\tau(t))\right),\ z_{ij}(t-\tau_1),\ f\left(z_{ij}(t-\tau_1)\right),\ z_{ij}(t-\tau_2),\ f\left(z_{ij}(t-\tau_2)\right),$$
$$\tfrac{1}{\tau_1}\int_{t-\tau_1}^{t} z_{ij}(s)ds,\ \tfrac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1} z_{ij}(s)ds,\ \tfrac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} z_{ij}(s)ds,\ \dot{z}_{ij}(t-\tau_1),\ \dot{z}_{ij}(t),\ z_{ij}(t-\sigma),\ \tfrac{1}{\sigma}\int_{t-\sigma}^{t} z_{ij}(s)ds\Big\}.$$

Then we have the following result.

**Theorem1** Under assumption $(\mathrm{H1})-(\mathrm{H2})$, for given scalars $\tau_1,\tau_2,\mu_1,\mu_2$ and $\sigma$ the networks (4) is global asymptotically synchronized if there exist real matrices $P>0,\ Q_k>0,\ S_k>0,\ Y_1>0, Y_2>0,\ R_1>0,\ \ R_2>0, X_1>0, X_2>0,\ (k=0,1,2,3)$ real positive diagonal matrices $\Lambda_1,\Lambda_2,\Lambda_3$ real matrix $D, M_1, K_1,\ \ M_2, K_2, Z_1, Z_2, B_1, B_2, \Theta_1$ and $\Theta_2$ of appropriate dimensions and $\varepsilon>0$ such that the following linear matrix inequalities hold for all $1\le i\le j\le N$:

$$\begin{bmatrix} \Pi_0\left(\tau(t),\dot{\tau}(t)\right)+\varepsilon\Pi_2^T\Pi_2 & \Pi_1^T \\ \Pi_1 & -\varepsilon I \end{bmatrix}_{\tau_1\le\tau(t)\le\tau_2,\,\mu_1\le\dot{\tau}(t)\le\mu_2}<0, \tag{5}$$

$$\begin{bmatrix} M_1 & K_1 \\ * & Y_2 \end{bmatrix}>0, \begin{bmatrix} M_2 & K_2 \\ * & X_2 \end{bmatrix}>0, \begin{bmatrix} \bar{R}_1 & D \\ * & \bar{R}_1 \end{bmatrix}>0, \begin{bmatrix} Z_1 & B_1 \\ * & R_2 \end{bmatrix}>0, \begin{bmatrix} Z_2 & B_2 \\ * & R_2 \end{bmatrix}>0,\ Z_1>Z_2, \tag{6}$$

where $\bar{R}_1=\mathrm{diag}\{R_1,3R_1\}$ and

$$\begin{cases} \Pi_0\left(\tau(t),\dot{\tau}(t)\right)= \Sigma_{1ij}+\Sigma_{2ij}+\Sigma_{3ij}+\Sigma_{4ij}+\Sigma_{5ij}+\Sigma_{6ij}+\Sigma_{7ij}, \\ \Pi_1=e_1^T\Theta_1H+e_{13}^T\Theta_2H, \qquad \Pi_2=-E_Ae_{14}+E_{W_1}e_2+E_{W_2}e_4+E_{\Gamma_1}e_1+E_{\Gamma_2}e_3+E_{\Gamma_3}\tau_1e_9+E_{\Gamma_3}\left(\tau(t)-\tau_1\right)e_{10}, \\ \Sigma_{1ij}=e_1^TPe_{13}+e_{13}{}^TPe_1+e_{13}^TQ_0e_{13}-e_{12}^TQ_0e_{12}+e_1^TS_0e_1-e_{14}^TS_0e_{14}, \qquad \Sigma_{2ij}=\left(1-\dot{\tau}(t)\right)\Upsilon_1+\Upsilon_2, \\ \Sigma_{3ij}=e_{13}{}^T\left(\tau_1^2Y_1+\tau_1^2Y_2\right)e_{13}-\varnothing_1^T\mathrm{diag}\{Y_1,3Y_1\}\varnothing_1+2\tau_1\left[K_1\left(e_1-e_9\right)+\left(e_1-e_9\right)^TK_1^T\right]+\tau_1^2M_1, \\ \Sigma_{4ij}=\left(\tau_2-\tau(t)\right)^2\left(Z_1-Z_2\right)+\left(\tau_2-\tau(t)\right)\Upsilon_3+\left(\tau(t)-\tau_1\right)\Upsilon_4+\Upsilon_5, \\ \Sigma_{5ij}=e_{13}{}^T\left(\sigma^2X_1+\sigma^2X_2\right)e_{13}-\varnothing_4^T\mathrm{diag}\{X_1,3X_1\}\varnothing_4+2\sigma\left[K_2\left(e_1-e_{15}\right)+\left(e_1-e_{15}\right)^TK_2^T\right]+\sigma^2M_2, \\ \Sigma_{6ij}=\Upsilon_6^T\Lambda_1\Upsilon_7+\Upsilon_7^T\Lambda_1\Upsilon_6+\Upsilon_8^T\Lambda_2\Upsilon_9+\Upsilon_9^T\Lambda_2\Upsilon_8+\Upsilon_{10}^T\Lambda_3\Upsilon_{11}+\Upsilon_{11}^T\Lambda_3\Upsilon_{10}, \qquad \Sigma_{7ij}=\Upsilon_{12}^T\Upsilon_{13}+\Upsilon_{13}^T\Upsilon_{12}, \end{cases} \tag{7}$$

with

$$\Upsilon_1=e_3{}^T\left(Q_3-Q_2\right)e_3+e_4{}^T\left(S_3-S_2\right)e_4,\quad \Upsilon_2=e_1{}^TQ_1e_1-e_7{}^TQ_3e_7+e_5{}^T\left(Q_2-Q_1\right)e_5+e_2{}^TS_1e_2-e_8{}^TS_3e_8+e_6{}^T\left(S_2-S_1\right)e_6,$$

$$\Upsilon_3=2B_1\left(e_3-e_{11}\right)+2\left(e_3-e_{11}\right)^TB_1^T+2B_2\left(e_5-e_3\right)+2\left(e_5-e_3\right)^TB_2^T, \qquad \Upsilon_4=2B_2\left(e_5-e_{10}\right)+2\left(e_5-e_{10}\right)^TB_2^T,$$

$$\Upsilon_5=(\tau_2-\tau_1)^2e_{12}^T\left(R_1+R_2\right)e_{12}+\tau_{21}^2Z_2+\varnothing_2^TD\varnothing_3+\varnothing_3^TD\varnothing_2+\varnothing_2^T\bar{R}_1\varnothing_2+\varnothing_3^T\bar{R}_1\varnothing_3, \qquad \Upsilon_6=e_2-L^-e_1,$$

$$\Upsilon_7=L^+e_1-e_2,\qquad \Upsilon_8=e_4-L^-e_3,\qquad \Upsilon_9=L^+e_3-e_4,\qquad \Upsilon_{10}=e_2-e_4-L^-e_1+L^-e_3,\qquad \Upsilon_{11}=L^+e_1-L^+e_3-e_2+e_4,$$

$$\Upsilon_{12}=e_1^T\Theta_1+e_{13}^T\Theta_2,\ \ \Upsilon_{13}=-e_{13}-Ae_{14}+W_1e_2+W_2e_4-Ng^{(1)}{}_{ij}\Gamma_1e_1-Ng^{(2)}{}_{ij}\Gamma_2e_3-\tau_1Ng^{(3)}{}_{ij}\Gamma_3e_9-\left(\tau(t)-\tau_1\right)Ng^{(3)}{}_{ij}\Gamma_3e_{10}.$$

**Proof:** Choose a LKF candidate as:

$$V(x_t)=\ \sum_{k=1}^{5}V_k(x_t), \tag{8}$$

where $V_1\left(x_t\right)=x^T(t)\,(U\otimes P)x(t)+\int_{t-\tau_1}^{t}\dot{x}^T(s)\,(U\otimes Q_0)\dot{x}(s)\,ds+\int_{t-\sigma}^{t}x^T(s)\,(U\otimes S_0)x(s)\,ds,$

$$\begin{aligned} V_2\left(x_t\right)=&\int_{t-\tau_1}^{t}\left[x^T(s)\,(U\otimes Q_1)x(s)+f^T\left(x(s)\right)(U\otimes S_1)f\left(x(s)\right)\right]ds \\ &+\int_{t-\tau(t)}^{t-\tau_1}\left[x^T(s)\,(U\otimes Q_2)x(s)+f^T\left(x(s)\right)(U\otimes S_2)f\left(x(s)\right)\right]ds \\ &+\int_{t-\tau_2}^{t-\tau(t)}\left[x^T(s)\,(U\otimes Q_3)x(s)+f^T\left(x(s)\right)(U\otimes S_3)f\left(x(s)\right)\right]ds, \end{aligned}$$

$$V_3\left(x_t\right)=\int_{t-\tau_1}^{t}\left[\tau_1\left(\tau_1-t+s\right)\dot{x}^T(s)\,(U\otimes Y_1)\dot{x}(s)+\left(\tau_1-t+s\right)^2\dot{x}^T(s)\,(U\otimes Y_2)\dot{x}(s)\right]ds,$$

$$V_4\left(x_t\right)=\int_{t-\tau_2}^{t-\tau_1}\left[\tau_{21}\left(\tau_1-t+s\right)\dot{x}^T(s)\,(U\otimes R_1)\dot{x}(s)+\left(\tau_1-t+s\right)^2\dot{x}^T(s)\,(U\otimes R_2)\dot{x}(s)\right]ds,$$

$$V_5\left(x_t\right)=\int_{t-\sigma}^{t}\left[\sigma\left(\sigma-t+s\right)\dot{x}^T(s)\,(U\otimes X_1)\dot{x}(s)+\left(\sigma-t+s\right)^2\dot{x}^T(s)\,(U\otimes X_2)\dot{x}(s)\right]ds.$$

Calculating the time derivative of $V\left(x_t\right)$ along the trajectories of (4) and by using Lemma (2)-(4), we have

$$\begin{aligned} \dot{V}_1(x_t)=&\,x^T(t)(U\otimes P)\dot{x}(t)+\dot{x}^T(t)(U\otimes P)x(t)+\dot{x}^T(t)(U\otimes Q_0)\dot{x}(t)+x^T(t)(U\otimes S_0)x(t) \\ &-\dot{x}^T(t-\tau_1)(U\otimes Q_0)\dot{x}(t-\tau_1)-x^T(t-\sigma)(U\otimes S_0)x(t-\sigma), \\ =&\sum_{1\le i<j\le N}^{N}\zeta_{ij}{}^T(t)\Sigma_{1ij}\,\zeta_{ij}(t), \end{aligned} \tag{9}$$

$$\dot{V}_2(x_t)\le\ \sum_{1\le i<j\le N}^{N}\zeta_{ij}{}^T(t)\Sigma_{2ij}\,\zeta_{ij}(t), \tag{10}$$

$$\begin{aligned} \dot{V}_3(x_t)=&\ \tau_1^2\dot{x}^T(t)(U\otimes(Y_1+Y_2))\,\dot{x}(t)\ -\int_{t-\tau_1}^{t}\tau_1\dot{x}^T(s)(U\otimes Y_1)\dot{x}(s)\,ds\ -\int_{t-\tau_1}^{t}2(\tau_1-t+s)\,\dot{x}^T(s)(U\otimes Y_2)\dot{x}(s)\,ds, \\ \le&\sum_{1\le i<j\le N}^{N}\zeta_{ij}{}^T(t)\Sigma_{3ij}\,\zeta_{ij}(t), \end{aligned} \tag{11}$$


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$$\dot{V}_4(x_t) = \tau_{21}^2 \dot{x}^T(t-\tau_1)(U\otimes(R_1+R_2))\dot{x}(t-\tau_1) - \int_{t-\tau_2}^{t-\tau_1} \tau_{21}\dot{x}^T(s)(U\otimes R_1)\dot{x}(s)ds - \int_{t-\tau_2}^{t-\tau_1} 2(\tau_2 - t + s)\dot{x}^T(s)(U\otimes R_2)\dot{x}(s)ds,$$

$$\leq \sum_{1\leq i<j\leq N}^{N} \zeta_{ij}^T(t)\Sigma_{4ij}\zeta_{ij}(t), \qquad (12)$$

$$\dot{V}_5(x_t) = \sigma^2 \dot{x}^T(t)(U\otimes(X_1+X_2))\dot{x}(t) - \int_{t-\sigma}^{t} \sigma\dot{x}^T(s)(U\otimes X_1)\dot{x}(s)\,ds - \int_{t-\sigma}^{t} 2(\sigma - t + s)\dot{x}^T(s)(U\otimes X_2)\dot{x}(s)\,ds,$$

$$\leq \sum_{1\leq i<j\leq N}^{N} \zeta_{ij}^T(t)\Sigma_{5ij}\zeta_{ij}(t), \qquad (13)$$

where $\bar{R}_1 = \mathrm{diag}\{R_1, 3R_1\}$ , $\varnothing_1 = \mathrm{col}\{e_1 - e_5, e_1 + e_5 - 2e_9\}$ , $\varnothing_2 = \mathrm{col}\{e_3 - e_7, e_3 + e_7 - 2e_{10}\}$ , $\varnothing_3 = \mathrm{col}\{e_5 - e_3, e_5 + e_3 - 2e_{10}\}$ , $\varnothing_4 = \mathrm{col}\{e_1 - e_{14}, e_1 + e_{14} - 2e_{15}\}$ , $\Sigma_{1ij}$, $\Sigma_{2ij}$, $\Sigma_{3ij}$, $\Sigma_{4ij}$ and $\Sigma_{5ij}$ are defined in (7).

From (3), it can be deduced that, for any $\lambda_{1k}\geq 0, \lambda_{2k}\geq 0, \lambda_{3k}\geq 0$, $k=1,2,...,n,$ we have

$$2[f_k(z_{ij}(t)) - l_k^- z_{ij}(t)]\,\lambda_{1k}\,[l_k^+ z_{ij}(t) - f_k(z_{ij}(t))] \geq 0,$$

$$2[f_k(z_{ij}(t-\tau(t))) - l_k^- z_{ij}(t-\tau(t))]\,\lambda_{2k}\,[l_k^+ z_{ij}(t-\tau(t)) - f_k(z_{ij}(t-\tau(t)))] \geq 0,$$

$$2[f_k(z_{ij}(t)) - f_k(z_{ij}(t-\tau(t))) - l_k^-(z_{ij}(t) - z_{ij}(t-\tau(t)))]\lambda_{3k}[l_k^+(z_{ij}(t) - z_{ij}(t-\tau(t))) - f_k(z_{ij}(t-\tau(t))) + f_k(z_{ij}(t))] \geq 0,$$

where $\Lambda_1 = \mathrm{diag}\{\lambda_{11},...,\lambda_{1n}\}$, $\Lambda_2 = \mathrm{diag}\{\lambda_{21},...,\lambda_{2n}\}$ and $\Lambda_3 = \mathrm{diag}\{\lambda_{31},...,\lambda_{3n}\}$, which imply

$$\sum_{1\leq i<j\leq N}^{N} \zeta_{ij}^T(t)\,\Sigma_{6ij}\,\zeta_{ij}(t) \geq 0, \qquad (14)$$

where $\Sigma_{6ij}$ is defined in (7). And, for any matrices $\Theta_1$ and $\Theta_2$ with appropriate dimensions, it is true that

$$0 = 2\left(x^T(t)(U\otimes\Theta_1) + \dot{x}^T(t)(U\otimes\Theta_2)\right)\Big[-\dot{x}(t) - \left(I_N\otimes(A+\Delta A(t))e_{14} + \left(I_N\otimes(W_1+\Delta W_1(t)\right)e_2 + \left(I_N\otimes(W_2+\Delta W_2(t)\right)e_4$$

$$+\left(G^{(1)}\otimes(\Gamma_1+\Delta\Gamma_1(t))\right)e_1 + \left(G^{(2)}\otimes(\Gamma_2+\Delta\Gamma_2(t))\right)e_3 + \left(G^{(3)}\otimes(\Gamma_3+\Delta\Gamma_3(t))\right)\left(\tau_1 e_9 + \left(\tau(t)-\tau_1\right)e_{10}\right)\Big]. \qquad (15)$$

So, we obtain

$$\dot{V}(t,x_t) \leq \sum_{1\leq i<j\leq N}^{N} \zeta_{ij}^T(t)\Big\{\Sigma_{1ij} + \Sigma_{2ij} + \Sigma_{3ij} + \Sigma_{4ij} + \Sigma_{5ij} + \Sigma_{6ij} + e_{13}^T\Theta\Big[-(A+\Delta A(t))e_{14} + (W_1+\Delta W_1(t))e_2$$

$$+(W_2+\Delta W_2(t))e_4 - Ng_{ij}^{(1)}\Gamma_1 e_1 - Ng_{ij}^{(2)}\Gamma_2 e_3 - \tau_1 N g_{ij}^{(3)}\Gamma_3 e_9 - \left(\tau(t)-\tau_1\right)N g_{ij}^{(3)}\Gamma_3 e_{10} - e_{13}\Big]\Big\}\zeta_{ij}(t). \quad (16)$$

Replacing $\Delta A(t), \Delta W_1(t), \Delta W_2(t), \Delta\Gamma_1(t), \Delta\Gamma_2(t)$, and $\Delta\Gamma_3(t)$ with $HF(t)E_A, HF(t)E_{W_1}, HF(t)E_{W_2}, HF(t)E_{\Gamma_1}$ $HF(t)E_{\Gamma_2}$ and $HF(t)E_{\Gamma_3}$ respectively, so we have

$$\dot{V}(x_t) \leq \sum_{1\leq i<j\leq N}^{N} \zeta_{ij}^T(t)\left[\Pi_0\left(\tau(t),\dot{\tau}(t)\right) + \Pi_1^T F(t)\Pi_2 + \Pi_2^T F(t)\Pi_1\right]\zeta_{ij}(t), \qquad (17)$$

where $\Pi_0\left(\tau(t),\dot{\tau}(t)\right)$, $\Pi_1$ and $\Pi_2$ are defined in (7). By Lemma 5, it can be deduced that $\varepsilon>0$ and

$$\dot{V}(x_t) \leq \sum_{1\leq i<j\leq N}^{N} \zeta_{ij}^T(t)\left[\Pi_0\left(\tau(t),\dot{\tau}(t)\right) + \varepsilon^{-1}\Pi_1^T\Pi_1 + \varepsilon\Pi_2^T\Pi_2\right]\zeta_{ij}(t). \qquad (18)$$

Applying the Schur complement Lemma, we know that $\Pi_0\left(\tau(t),\dot{\tau}(t)\right) + \varepsilon^{-1}\Pi_1^T\Pi_1 + \varepsilon\Pi_2^T\Pi_2 < 0$, is equivalent to (5). Thus (5) and (6) hold, implies that $\dot{V}(x_t) < 0$, we know that $x^T(t)(U\otimes P)x(t)$ is bounded function and $\lim_{t\to\infty}\|z_{ij}\| = \lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0.$ By Lyapunove theorem and Definition 1, the neural networks (4) are globally robustly synchronized. □

## III. MAIN RESULT

In this section, we present a numerical example to illustrate the effectiveness and the reduced conservatism of our result. Consider the dynamical system consisting of model (1), choose the following matrix:

$$A = \begin{bmatrix} 3.5 & 0 \\ 0 & 1.5 \end{bmatrix}, W_1 = \begin{bmatrix} -1 & 0.4 \\ -0.2 & -0.1 \end{bmatrix}, W_2 = \begin{bmatrix} 0.4 & -0.6 \\ 0.2 & 0.4 \end{bmatrix}, \Gamma_1 = \Gamma_2 = \Gamma_3 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \; g^{(1)} = g^{(2)} = g^{(3)} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, E_A = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix},$$

$$E_{W_1} = \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.4 \end{bmatrix}, E_{W_2} = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}, E_{\Gamma_1} = E_{\Gamma_2} = E_{\Gamma_3} = \begin{bmatrix} 0.1 & 0 \\ -0.1 & 0.1 \end{bmatrix}, H = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, F(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & \cos(t) \end{bmatrix}, \; 0.3 \leq \tau(t) \leq 1,$$

$-0.5 \leq \dot{\tau}(t) \leq 0.5$ and $\sigma = 0.2$. The neural activation functions as follows: $f_1(x_1(t)) = 0.4\tanh(-x_1)$, $f_2(x_2(t)) = 0.6\tanh(-x_2)$. By applying the MATLAB LMI Toolbox in Theorem1, it yields the following feasible solutions:


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$$P=\begin{bmatrix}0.0347 & 0.0024\\ 0.0024 & 0.0925\end{bmatrix}, Q_0=\begin{bmatrix}0.0154 & 0.0026\\ 0.0026 & 0.0647\end{bmatrix}, S_0=\begin{bmatrix}0.0906 & 0.0462\\ 0.0462 & 0.2326\end{bmatrix}, S_1=\begin{bmatrix}0.3328 & 0.0195\\ 0.0195 & 0.2772\end{bmatrix}, S_2=\begin{bmatrix}0.1061 & 0.0138\\ 0.0138 & 0.0875\end{bmatrix}, S_3=\begin{bmatrix}0.2959 & -0.0033\\ -0.0033 & 0.2527\end{bmatrix},$$

$$Q_1=\begin{bmatrix}0.1459 & 0.0161 & -0.1223 & -0.0068\\ 0.0161 & 0.3658 & -0.0184 & -0.2557\\ -0.1223 & -0.0184 & 0.3038 & 0.0011\\ -0.0068 & -0.2557 & 0.0011 & 0.4078\end{bmatrix}, \quad Q_2=\begin{bmatrix}0.0895 & 0.0027 & 0.0041 & -0.0014\\ 0.0027 & 0.1914 & -0.0066 & -0.0150\\ 0.0041 & -0.0066 & 0.1688 & -0.0010\\ -0.0014 & -0.0150 & -0.0010 & 0.2319\end{bmatrix}, \quad Q_3=\begin{bmatrix}0.0833 & 0.0051 & 0.0124 & -0.0010\\ 0.0051 & 0.1887 & 0.0009 & 0.0143\\ 0.0124 & 0.0009 & 0.0925 & 0.0037\\ -0.0010 & 0.0143 & 0.0037 & 0.1196\end{bmatrix},$$

$$Y_1=\begin{bmatrix}0.0126 & 0.0012\\ 0.0012 & 0.0384\end{bmatrix}, Y_2=\begin{bmatrix}0.0118 & 0.0022\\ 0.0022 & 0.0593\end{bmatrix}, R_1=\begin{bmatrix}0.0168 & 0.0009\\ 0.0009 & 0.0394\end{bmatrix}, R_2=\begin{bmatrix}0.0054 & 0.0028\\ 0.0028 & 0.0479\end{bmatrix}, X_1=\begin{bmatrix}0.3673 & -0.0111\\ -0.0111 & 0.2706\end{bmatrix}, X_2=\begin{bmatrix}0.6371 & 0.0048\\ 0.0048 & 0.8024\end{bmatrix},$$

$$\Theta_1=\begin{bmatrix}0.1091 & 0.0067\\ 0.0067 & 0.2738\end{bmatrix}, \quad \Theta_2=\begin{bmatrix}0.0238 & 0.0043\\ 0.0043 & 0.1062\end{bmatrix}, \quad \Lambda_1=\begin{bmatrix}0.4670 & 0.0000\\ 0.0000 & 0.3147\end{bmatrix}, \quad \Lambda_2=\begin{bmatrix}0.3086 & 0.0000\\ 0.0000 & 0.3169\end{bmatrix}, \quad \Lambda_3=\begin{bmatrix}0.0748 & 0.0000\\ 0.0000 & 0.0935\end{bmatrix}, \qquad \varepsilon=0.1169.$$

The simulation results for the synchronization errors, $z_{i1}(t)=x_i(t)-x_1(t),\ (i=2,3)$ of networks (1) are show in Figs.1. The figures show that the networks with the errors converge to zero for given initial values by $x_1(0)=[0.3\ \ -0.3]^T$, $x_2(0)=[0.4\ \ -0.35]^T$ and $x_3(0)=[0.1\ \ -0.45]^T$.

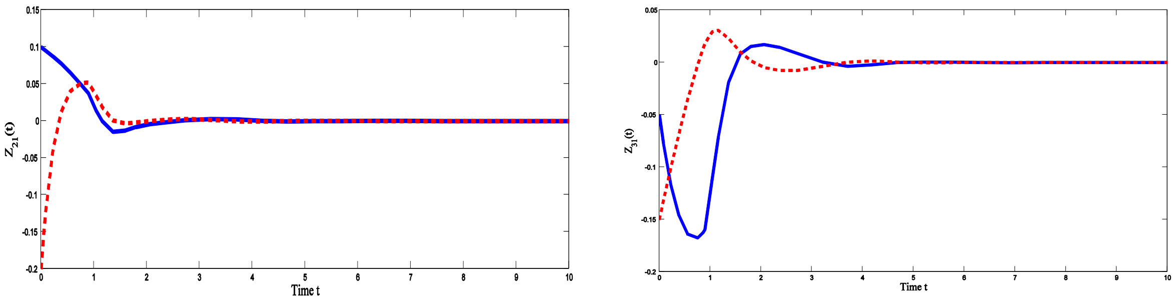


**Fig. 1 Synchronization errors of networks** $z_{i1}(t)=x_i(t)-x_1(t),\ (i=2,3)$

## IV. CONCLUSIONS

In this paper, the robust synchronization analyses of coupled neural networks with interval time-varying delays and leakage delay have been proposed. By constructing a set of improved LKF and Kronecker product properties, sufficient conditions for guaranteeing asymptotic synchronization for the concerned networks have been derived in terms of LMI. A numerical example illustrates the effectiveness of the estimated result.

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