Subspace Homotopy Methods for Solving Nonlinear Equations

S.R.Saratha, Vijeta Iyer, R.Manju

***Abstract:** The concept of Newton subspace homotopy method has been introduced in this paper. As Newton – Homotopy method can't be used for all the functions in topological spaces. So an algorithm has been developed by finding a subspace of the given space for which the subspace homotopy function has been found.*

***Keywords:** Homotopy, Newton-Raphson Method, Subspace Homotopy, Matlab 2010 AMS Subject Classification: 55P10, 55P35, 55P99.*

I. INTRODUCTION

Newton method gives accurate and finest numerical results than other numerical methods. The convergence rate of Newton Raphson method is Quadratic (ie. 2). Because of this reason, it converges better than other numerical methods. For finding the solution of $u(x) = 0;\ x \in \Re;\ u(x)$ may be polynomials, ordinary or partial differential equations. In the Newton Raphson method formula, the term should be $u'(x_i) \neq 0$. If $u'(x_i) = 0$, then the system fails and usage of derivative is very difficult to solve.

The disadvantage of the Newton's method is fruitfully overcome by a universal numerical method called subspace homotopy. This method is used in many areas of engineering fields. Giving importance to develop the iterative methods for solving nonlinear equations [1]. Homotopy can guarantee to converge by certain path if suitable auxiliary homotopy function is chosen.

For every function, it is impossible to find a Homotopy function. For such functions, a subspace in which the homotopy function can be obtained is considered and then the roots can be found easily.

The developed algorithms of subspace homotopy equations with the help of Newton method to find the roots of the polynomials. The efficacy of these methods is demonstrated and relation of convergence and number of iterations are studied.

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II. PRELIMINARIES

Definition 2.1: [2] Let E and F be the topological spaces and B be the subspace of E and let continuous functions u and v be from E to F. Then u and v are subspace homotopic, if a function G from $B \times I$ to F denoted by $G(e,s) = (1-s)u(e) + sv(e)$ exist, such that (i) G is continuous (ii) $G(e,0) = f(e)$ and $G(e,1) = g(e)$ for all e in B and s in I = [0,1]. In this case we write $[u \approx v]_B$.

Example 2.1: [2] s E= [0, 3] and F= R and it is defined by $u(e) = 1 + e^2(e-3)^2$ and $v(e) = 1$. Let B = [0, 1] be the subspace of E. Then u and v are subspace homotopic defined by the map $G(e,s) = 1 + (1-s)e^2(e-2)^2$ for all e in B.

The **c**oncept of Subspace homotopy and Homotopy are independent by using the following examples.

Example 2.2: [2] Let E and F be topological spaces and B be the subspace of E and u and v be any two continuous maps of a space E= [0, 2] into R and it is defined by $u(e) = 1 + \frac{e^2}{(e-2)^2}$ and $v(e) = 1$. Consider B = [0, 1] as the subspace of E. Then u and v are not homotopic but subspace homotopic defined by the map $G(e,s) = 1 + (1-s)\frac{e^2}{(e-2)^2}$.

Example 2.3: [2] Let E and F be topological spaces and B be the subspace of E and u and v be any two maps of a space R into R and it is defined by $u(e) = e^2 + 1$ and $v(e) = \cos(4e) - 1$. Let B = Z be the subspace of E. Then u and v are homotopic but not subspace homotopic defined by the map $G(e,s) = (1-s)u(e) + sv(e)$.

Theorem 2.2: [2] Let E be a normal space and B be a closed subspace of E. Then any subspace homotopy on B into R may be extended to homotopy on E.

Proof: By using Tietze extension theorem, we will prove this result.

Definition 2.3: [2] Let E be a topological space and B a subspace of E. If $u:[0,1] \to E$ is continuous such that $u(0) = e_0$ and $u(1) = e_1$, $x_0, x_1 \in B$. Then $v = u_{/B}$ is a subspace path from e_0 to e_1.

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Definition 2.4: [2] Let E be a topological space and B be the subspace of E. Two paths u and v mapping the interval $I = [0,1]$ into E are known as subspace path homotopic on B if the starting point $e_0 \in B$ and ending point $e_1 \in B$ are same and if there exist a continuous function $G : I \times I \to B$ where $G(s,0) = u(s)$ and $G(s,1) = v(s)$, $G(0,t) = e_0$ and $G(1,t) = e_1$ for each $s \in I$ and $t \in I$. Then G is a subspace path homotopic between u and v and is denoted by $(u \approx_p v)_B$.

Example 2.4: [2] Let E be a topological space and B be the subspace of E where E= R^2 and B= the punctured plane $R^2 - \{0\}$. The paths $u : I \to E, v : I \to E, w : I \to E$ defined by

$$u(s) = (\cos \pi s, \sin \pi s), v(s) = (\cos \pi s, 2\sin \pi s).$$

$$w(s) = (\cos \pi s, -\sin \pi s)$$

Then u and v are subspace path homotopy and path homotopy but v and w and u and w are path homotopy but not subspace path homotopy.

Theorem 2.5: [2] If B is a subspace of E, then the relation subspace homotopic $(\approx)_B$ is an equivalence relation.

Definition 2.6: [2] If the paths u from e_0 .to e_1 .and v from e_1 .to e_2 . in B. Then w= u*v of u and v given by the equation $w(s) = \begin{cases} u(2s) & for \quad s \in [0,1/2] \\ v(2s-1) & for \quad s \in [1/2,1]. \end{cases}$

III. DESIGN OF THE PROJECT

Definition 3.1: A polynomial $u(e)$ of degree p is defined as

$$u(e) = a_0 e^0 + a_1 e^1 + a_2 e^2 + \ldots\ldots + a_p e^p. \quad (1)$$

Define the convex subspace homotopy for the function

$$G(e,s) = (1-s)u(e) + sv(e), \quad (2)$$

such that (i) continuous (ii) $G(e,0) = u(e)$ and $G(e,1) = v(e)$ for all e in B and s in [0, 1].

There are four ways to identify a begin system of the subspace homotopy [4] such as

(i) The fixed point homotopy:

$$u(e) = e - e_0,$$

$$G(e,s) = (1-s)(e - e_0) + sv(e) = 0 \quad (3)$$

Where e_0 is an initial approximation of Eq(2).

(ii) The Newton – homotopy

$$u(e) = q(e) - q(e_0),$$

$$G(e,s) = (1-s)(q(e) - q(e_0)) + sv(e) = 0 \quad (4)$$

(iii) The start – system Newton Homotopy:

$$u(e) = e^n - C,$$

$$G(e,s) = (1-s)(e^n - C) + sv(e) = 0 \quad (5)$$

where the highest power of e is n,
C is a constant and the roots of u(e) = 0 can be obtained.[3]

(iv) The begin – system Newton subspace homotopy:

$$u(e) = e^n - C$$

$$G(e,s) = (1-s)(e^n - C) + sv(e) = 0 \quad (6)$$

where the highest power of e is n, C is a constant and the zeros of u(e) can be obtained.

By using Newton – homotopy, it does not guarantee convergence as well as use of start – system Newton homotopy cannot find all forms of functions in the previous section examples but solve the above examples by using begin-system Newton subspace homotopy method. So, the next section will bring forward begin – system Newton subspace homotopy method by using following example:

Example 3.1: Let E and F be topological spaces and B be the subspace of E and u and v be two maps of a space E= R into R and defined by $u(e) = e^2 - 9 + \frac{e^2}{(e-5)^2}$ and $v(e) = e^2$. Let B = [-3,3] be the subspace of E. Then u and v are subspace homotopic but not homotopic defined by the map $G(e,s) = s(e^2 - 9) + (1-s)\frac{e^2}{(e-5)^2}$.

Where,

s is an embedded parameter and s is in the interval [0, 1];

$u(e)$ is the begin system;

$v(e) = q(e)$ is the end system;

$G(e,0) = u(e)$ and $G(e,1) = v(e)$ for all e in B and s in in the interval [0,1].

The Newton – subspace homotopy using begin system algorithm for the previous example using MATLAB has been derived. The iterations will follow the following Newton – subspace homotopy algorithms.

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Function v(e)=q(e)=0; = 'end system' and $G(e,s) = (1-s)u(e) + sv(e)$	Root(s) (Classical Newton Raphson Method) iteration,i	Root(s) using Newton Subspace Homotopy method; f(e)=0 ='begin-system' $s \to G(e,s)$	f(x)=0 and Initial point x_0	Iteration using NSHbs,i
$v_1(e) = (e^2 - 9) + \left(\frac{e^2}{(e-5)^2}\right)$ and $G(e,s) = (e^2 - 9) + s\left(\frac{e^2}{(e-5)^2}\right)$	-2.9767,2.7429 (5) (4)	0.0 $\to$ 3= e_0 0.2 - 2.9322 0.4 - 2.8753 0.6 - 2.8260 0.8 - 2.7822 1 - 2.7429	$e^2 - 9$ and 3,-3	1 2 2 2 3 3
$v_2(x) = (e^4 - 16) + \left(\frac{e^2}{(e-3)^2}\right)$ and $G(e,s) = (e^4 - 16) + s\left(\frac{e^2}{(e-3)^2}\right)$	-1.9950,1.8996 (4) (5)	0.0 $\to$ 2= x_0 0.2 - 1.9866 0.4 - 1.9203 0.6 - 1.9125 0.8 - 1.9056 1 - 1.8996	$(e^4 - 16)$ and 2,-2	1 2 2 2 2 2

Table 1: Numerical Solutions of existing Classical Newton – Raphson method with proposed method Newton – Subspace Homotopy using begin-system (NSHbs) using MATLAB R2017

Algorithm 3.1: Newton - subspace homotopy using begin-system

Step1: Find the Interval $B = [a,b]$ be the subset of E for which G is Subspace Homotopy.

Step2: Analyze $v(e) = q(e) = 0$.

Step3: Choose $u(e)$, such that $u(e) = e^2 - 9$.

Step4: Choose the initial value, e_0, by setting $u(e) = e^n - C = 0$ such as

- Restart;

  $v := x \to (x^2 - 9) + \frac{x^2}{(x-5)^2}$;

  $u := x \to x^2 - 9$; *solve(v(e));*
- NSH: =solve(u(e));

Step5: Simplify $G(e,s) = (1-s)(e^n - C) + sv(e) = 0$ such as

- $SF := e \to (1-s)u(e) + sv(e)$;simplify $(G(e,s))$;

Step6: Iterate $G(e,s) = (1-s)(e^n - C) + sv(e) = 0$ where $s \in [0,1]$ e. g: 0, 0.2, 0.4, 0.6, 0.8, 1 by using the classical Newton Raphson method

- $SFD := D(F); simplify \ (SFD(e,t))$;
- $newtsh := e \to evalf\left(e - \frac{FS(e)}{SFD(e)}\right)$

IV . COMPARISON WITH NEWTON RAPHSON METHOD

Few examples are developed to illustrate the convergence of the iterative methods such as Newton-Raphson and the begin-system subspace homotopy. By using the following examples, the following results are obtained as shown in Table 1:

(a) $v_1(e) = (e^2 - 9) + \left(\frac{e^2}{(e-5)^2}\right)$, E=R, B = [-3,3]

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In this example, E is considered then it fails at e=5. Instead of this subspace B = [-3,3] is considered and will be converged.

(b) $v_2(x) = (e^4 - 16) + \left(\dfrac{e^2}{(e-3)^2} \right)$, E=R, B=[-2,2]

V. CONCLUSION

Newton subspace Homotopy using begin-system method converges better than classical Newton-Raphson method. Computing time is also reduced. Sometimes when the whole space is considered, the system fails. In this situation the Newton subspace homotopy begin system is useful.

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