Function-E-Chainble Sets in Bitopological Space

Vijeta Iyer, S R Saratha, V Sudha

***Abstract:** Inthis paper,the concept of function$-\epsilon$ $-$chainabilitybetween two sets in bitopological spaces using continuous function has been introduced which is the extension of function-ε-chain between two points of the bitopological space. A characterization of function$-\epsilon-$chainability betweentwo sets has been discussed in respect of function $-\epsilon-$chains between their points. Also, some results of [1] have been generalized for bitopological spaces.Subject Classification: AMS (2000):54A99*

***Keywords:** Simple chain, function -ϵ-chain between points in bitopological space, function - ϵ-chain between sets in bitopological space.*

I. INTRODUCTION

Throughout the paper,(X, τ_1, τ_2)will be considered as bitopological space with two topologies$\tau_1 and\, \tau_2$. Also $\tau = \tau_1 \cap \tau_2$and $f : (X, \tau) \to [0, \infty)$ will be referred to as a real valued, non-constant and continuous function unless mentioned otherwise.

II. DEFINITIONS

Let A be any subset of the bitopological space X. For $\epsilon >$ 0,

define$U_{bi-f\,\epsilon}(A) = \{ x \in X : |\, f(x) - f(A)\,| < \epsilon \}$,
where

$$|\, f(x) - f(A)\,| = inf\, \{\, |\, f(x) - f(a)\,| : a \in A\}$$

A.Definition

Let A be any subset of X.Then $bi-f-$diameter of A,denoted by $\delta_{bi-f}(A)$is defined as$sup\{\, |\, f(x) - f(a)\,| : x, a \in A\,\}$.

B. Definition

Consider$A\, and\, B$as two subsets ofthe space X. Then$bi-f-$distance between the two sets$A$ and Bdenoted by$d_{bi-f}(A, B)$is defined as

$inf\, \{|\, f(a) - f(b)\,| : a \in A, b \in B\}$.

C. Remark

$$U_{bi-f\,\epsilon}(A) = \{x : d_{bi-f}(x, A) < \epsilon\}$$

D. Definition

A bitopological space (X, τ_1, τ_2) is said to be function$-f-\epsilon-$ chainable if for $\epsilon > 0$ there is acontinuous and non-constant function $f : (X, \tau) \to [0, \infty)$, where $\tau = \tau_1 \cap \tau_2$ such that for every pair of points $x, y \in X$ there is a sequence

$x = x_0, x_1, x_2, \dots, x_n = y$of points in X with $|\, f(x_i) - f(x_{i-1})\,| < \epsilon$; $1 \le i \le n$

E. Definition

Let (X, τ_1, τ_2) be a bitopological space and let there exist a continuous, non –constant function$f$ from $(X, \tau_1 \cap \tau_2)$ to $[0, \infty)$ such that the space$X$ is function$-f-\epsilon-$chainable for each positive ε. Then the bitopological space (X, τ_1, τ_2) is said to be function – $bi-f-$chainable.

F. Definition

Let$A\, and\, B$ be any two subsets of X.A function– $bi-f-\epsilon-$chain of length n from set A to set$B$ is a finite sequence of subsets of Xsay $A_0, A_1, A_2, \dots, A_n$with $A = A_0$, $A_n = B, A_{i-1} \subset U_{bi-f\,\epsilon}(A_i)andA_i \subset U_{bi-f\,\epsilon}(A_{i-1})$. If function $-f-\epsilon-$chain exist betweentwo sets$A$ and B we say $< A, B >$ is function– $bi-f-\epsilon-$chainable and $< A, B >$ is function – $bi-f-$chainable if it is function$-bi-f-\epsilon-$chainable for every $\epsilon > 0$.

By using induction, construct the set $U^n_{bi-f\,\epsilon}(A)$ for each positive non zero integer n as follows:

For $n = 1$ set$U^1_{bi-f\,\epsilon}(A) = U_{bi-f\,\epsilon}(A)$

and for each $n \geq 2$ set$U^n_{bi-f\,\epsilon}(A) = U_{bi-f\,\epsilon}(U^{n-1}_{bi-f\,\varepsilon}(A))$.

Then the following results are observed:

(1)$U^n_{bi-f\,\epsilon}(A) \subset U^{n+1}_{bi-f\,\epsilon}(A)$

(2)$U^n_{bi-f\epsilon}(A) \subset U_{bi-f\,n\epsilon}(A)$

We define$\Phi_{bi-f\epsilon}(< A, B >)$as the length of function – $bi-f-\epsilon-$chain between two sets$A$ and B which is shortest.

G. Example of function – bi $-$f $-\epsilon-$chainable sets

Let (X, τ_1, τ_2) be a bitopological space with $\tau_1$as odd even topology generated by $P = \{\{1,2\}, \{3,4\}, \{5,6\}, \dots\}$ and τ_2 as a discrete topology. Also let$\tau = \tau_1 \cap \tau_2$. Consider $f : X \to [0, \infty)$ define by $f(2k) = k$, $f(2k-1) = k$ which is a continuous function. Let $A = \{1,2\}$, $B = \{3,4\}$and$\epsilon = 1.2$ then $U_{bi-f\,\epsilon}(A) = \{1,2,3,4\}$and $U_{bi-f\epsilon}(B) = \{1,2,3,4,5,6\}$ or $A \subset U_{bi-f\,\epsilon}(B)\, and\, B \subset U_{bi-f\,\epsilon}(A)\, or\, A = A_0, A_1 = B$. Hence $< A, B >$ is $bi-f-\varepsilon-$chainable for given$\epsilon = 1.2$

III. THEOREMS

A. Some obvious results.

a. Result

Let A and B be two subsets of X, then

a. $d_{bi-f}(A, B) < \epsilon$ if $B \cap U_{bi-f\,\epsilon}(A) \neq \emptyset$
b. $d_{bi-f}(A, B) < \epsilon$ if $A \cap U_{bi-f\epsilon}(B) \neq \emptyset$
c. $A \subset U_{bi-f\,\epsilon}(A)$
d. $U_{bi-f\,\epsilon}(A) \subset U_{bi-f\epsilon}(B)$ if $A \subset B$
e. $U_{bi-f\,\epsilon}(A) \cup U_{bi-f\,\varepsilon}(B) = U_{bi-f\,\epsilon}(A \cup B)$
f. $U_{bi-f\,\epsilon}(A \cap B) \subseteq U_{bi-f\epsilon}(A) \cap U_{bi-f\,\varepsilon}(B)$

Revised Version Manuscript Received on 25 November, 2018.

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b. Result

If $< A, B >$and $< C, D >$ are$bi-f$ $-$chainable sets then $< A \cup C , B \cup D >$ is also $bi-f$ $-$chainable whereA, B, C, D are subsets of X.

B. Theorem

Let (X, τ_1, τ_2) be a bitopological space and $A \subset X$ then

$$A \subseteq \bigcap_{\varepsilon>0} U_{bi-f\,\epsilon}(A) = \overline{A}$$

Proof: As A $\subset U_{bi-f\epsilon}(A)$, $\epsilon > 0$ then A $\subseteq \cap\, U_{bi-f\,\epsilon}(A)$

Let $x \in \overline{A}$then the image$f(x) \in f(\overline{A}) \subset \overline{f(A)}$

equivently there exists $y \in A$ such that $|f(x) - f(y)| < \epsilon$

equivently,$x \in U_{bi-f\,\epsilon}(A), \ \forall\ \varepsilon > 0$ or $\overline{A} \subset \bigcap_{\varepsilon>0} U_{bi-f\,\epsilon}(A)$

Consider that $\overline{A} \nsubseteq \cap\, U_{bi-f\,\epsilon}(A)$

equivently there exist $x \in \cap\, U_{bi-f\,\epsilon}(A)$such that $x \notin \overline{A}$ equivently,there exist $x \in U_{bi-f\epsilon}(A), \forall \epsilon > 0$such that $x \notin \overline{A}$and therefore$x \notin A$

equivently, $|f(x) - f(A)| \neq 0$or$|f(x) - f(A)| = \epsilon'$for any real $\epsilon' > 0$.

equivently$x \notin U_{bi-f\,\epsilon}(A)$ for $\epsilon < \epsilon'$.

This is a contradiction to the fact that $x \in U_{bi-f\,\epsilon}(A)\ \forall \epsilon > 0$

Thus$\overline{A} = \cap_{\epsilon>0} U_{bi-f\,\epsilon}(A)$.

a.Collolary

A set$A$ is closed if and only if

$$A = \bigcap_{\epsilon>0} V_{bi-f\,\epsilon}(A).$$

Next theorem establishes the characterization of thefunction – bi $-f-\epsilon-$chainable sets with regard to function – bi $-f-\epsilon-$chains between the points and sequence.

C. Theorem

Let $A$and Bbe two subsets of Xthen$< A, B >$is function – bi $-f-\epsilon-$ chainable if there exist afunction – bi $-f-\epsilon-$ chain from each and everyelement of set$A$ to some element of set$B$ and vice-versa.The converseis also true.

Proof: The necessary part: As $< A, B >$is function – bi $-f-\epsilon-$chainable sets, so there exists a sequence $A_0, A_1, \dots, A_n$of subsets of X with $A = A_0, A_n = B$, $A_i \subset U_{bi-f\,\epsilon}(A_{i-1})$ and $A_{i-1} \subset U_{bi-f\,\varepsilon}(A_i)$;$1 \le i \le n$. Let $x$be any arbitrary element of A. Then$x \in A$or $x \in U_{f\epsilon}(A_1)or$ $|f(x) - f(x_1)| < \epsilon$ for some $x_1 \in A_1$.Again as$x_1 \in A_1$ then $|f(x_1) - f(x_2)| < \epsilon$for some$x_2 \in A_2$.Continuing the same process n times we get a sequence$x = x_0, x_1, x_2, \dots, x_n = y \in B$with $|f(x_i) - f(x_{i-1})| < \epsilon$; $1 \le i \le n$ and $x_i \in A_i$,provingthe existence of afunction$-f-\epsilon-$chain from x to y. In similiar way we can get a function$-f-\epsilon-$chain from each and every element ofthe set$B$ to someelement of set$A$.

The sufficient part:Consider there is a function$-f-\epsilon-$chain from each and every element of A to some element of B and vice-versa. Let $A_1 = \{y \in X: |f(y) - f(x)| < \epsilon$ for some $x \in A$ and $x \neq y\}$.Clearly A_1 is non empty and $A_1 \subset U_{f\,\epsilon}(A)$. Next we prove that$A \subset U_{bi-f\,\epsilon}(A_1)$. If $x \in A$ then a sequenceof points $x = x_0, x_1, x_2 \dots, x_n = y \in B$exist with $|f(x) - f(x_1)| < \epsilon$ or $x_1 \in A_1$ then $|f(x) - f(A_1)| < \epsilon$or equivalently $x \in U_{bi-f\,\epsilon}(A_1)$ or $A \subseteq U_{bi-f\,\epsilon}(A_1)$.

Nextconsider $A_2 = \{y \in X: |f(y) - f(x)| < \epsilon\ for\ some\ x \in A_1\ and\ x \neq y\}$. Clearly A_2 is non empty set , $A_2 \subset V_{f\,\epsilon}(A_1)$and it can be proved as above that $A_1 \subset V_{bi-f\,\epsilon}(A_2)$. Continuing the same process$n$times we get a sequence $A = A_0, A_1, \dots, A_n = B$ of subsets of X, such that $< A, B >$ is function – bi $-f-\epsilon-$chainable sets.

For next few theorems proofs are omitted as they are direct and similar to proofs in [1].

D. Theorem

Let$A$and Bbe two subsets of X. Then if $\delta_{bi-f}(A \cup B) \le \epsilon$, $< A, B >$ is function – bi $-f-\epsilon-$chainable.

E. Theorem

Let $< \epsilon_n >$ be monotonically increasing sequence of positive real numbers converging to an arbitrary real number $\epsilon > 0$. Then $< A, B >$ is function – bi $-f-\epsilon$ – chainableif and only if there exists a subsequence $< \epsilon_{n_k} >$of the sequence$< \epsilon_n >$ such that $< A, B >$ is function – bi $-f-\epsilon_{n_k}$ – chainablefor each natural number k.

F. Theorem

Let A and B be two subsets of X. If $A \cup B$ is connected and $\epsilon > max\,\{\delta_{bi-f}(A), \delta_{bi-f}(B)\}$ then $< A, B >$is function – bi $-f-\epsilon-$chainable sets.

G. Theorem

Let A and B be two subsets of $X$and $\epsilon > max\,\{\delta_{bi-f}(A), \delta_{bi-f}(B), d_{bi-f}(A, B)\}$then sets A and B is function$-bi-f-\epsilon-$chainable sets and $\Phi_{bi-f\epsilon}(<A, B>=2$.

H. Theorem

(X, τ_1, τ_2) is function$-f-\epsilon-$chainable bitopological spaceiff$< A, B >$ is function$-bi-f-\epsilon-$chainable for every pair of subsets A, B of X.

I. Theorem

Let A and B be two subsets of X. Then $\overline{A} = \overline{B}iff< A, B >$ is function – bi $-f-$chainable and $\Phi_{bi-f\,\epsilon}(< A, B >) = 1$.

J. Theorem

If $U^n_{bi-f\,\epsilon}(A) \subset B \subseteq U^{n+1}_{bi-f\,\epsilon}(A)$,then $< A, B >$is function – bi $-f-\epsilon-$chainable and $\Phi_{bi-f\,\epsilon}(< A, B >) = n + 1$.

K. Theorem

Let X be function$-f-\epsilon-$chainable bitopological space. Then a relation $\approx$is an equivalence relation on X defined as $< A, B > \approx < C, D >$ iff $\Phi_{bi-f\,\epsilon}(< A, B >) = \Phi_{bi-f\,\epsilon}(< C,\ D >)$on X, and$X$is partitioned into disjoint equivalence classes denoted by $<\widehat{A, B}>$,$<\widehat{C, D}>$.

L. Theorem

Consider a simple chain$\{A = A_0, A_1, A_2, \dots, A_n = B\}$ then $< A, B >$ is function$-bi-f-2\epsilon-$chainablesets where $\epsilon > max\,\{\delta_f(A), \delta_f(A_1), \delta_f(A_2), \ldots, \delta_f(B)\}$.

Proof : Let $x \in A,\ \ y \in (A \cap A_1),\ \ z \in A_1$

then $|f(x) - f(y)| < \epsilon$ and $|f(y) - f(z)| < \epsilon$ or $|f(x) - f(z)| < 2\epsilon$ or$|f(x) - f(A_1)| < 2\epsilon$

or $x \in U_{bi-f\ 2\epsilon}(A_1) \Longrightarrow A \subseteq U_{bi-f\ 2\epsilon}(A_1)$.

then $\inf_{x\in A}|f(x) - f(z)| < 2\epsilon \Longrightarrow |f(z) - f(A)| < 2\epsilon$

$\Longrightarrow z \in U_{bi-f\ 2\epsilon}(A)$ or $A \subset U_{bi-f\ 2\epsilon}(A_1)andA_1 \subset U_{bi-f\ 2\epsilon}(A)$.

Same way, $A_1 \subset U_{bi-f\ \epsilon}(A_2)$, $A_2 \subset U_{bi-f\ \epsilon}(A_1)$, …,$A_{n-1} \subset U_{bi-f\ \epsilon}(A_n)$ and $A_n \subset U_{bi-f\ \epsilon}(A_{n-1})$.

Hence the sets $A_1, A_2, \ldots, A_{n-1}$ forms a function – bi $-f-\epsilon-$chain from set$A$ to set$B$or equivalently $< A, B >$ is function$-f-2\epsilon-$chainable.

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