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Balancing Double Inverted Pendulum on A cart by Linearization Technique

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Abstract*: Double Inverted Pendulum on a cart (DIPC) is a classic problem in dynamics and control theory and is widely used in control laboratories to demonstrate the effectiveness of control systems as well as a benchmark for testing control algorithms. It is suitable to investigate and verify different control methods for dynamic systems with higher order nonlinearities. A controller is proposed to swing a double inverted pendulum to an unstable upright inverted position and stabilize around the point by using Linear Quadratic Regulator (LQR) technique. Linearization form of nonlinear system is obtained by Jacobian with proper cost function and the modeling of it is accomplished with the help of Euler – Lagrangian equation derived by specifying Lagrangian, difference between kinetic and potential energy of DIPC. Simulation results are retrieved by MATLAB.

Index Terms*: Double Inverted Pendulum on a Cart, Jacobian, Lagrangian, LQR.

I. INTRODUCTION

The Linear Double Inverted Pendulum module is perfect to present transitional and propelled control ideas, taking the fantastic single transformed pendulum test to the following level of unpredictability. In a twofold reversed pendulum framework examined here, two pendulums are joined and the easier pendulum is depended on a cart. Since the upright state of the two pendulums is a flimsy harmony point, the two pendulums will tumble down without control if one of the pendulums does not remained up. In this paper an under actuated nonlinear unstable plant double inverted pendulum on a cart is used as a test bench.

Stabilization of DIPC to an unstable equilibrium position by various control strategies such as PID controller, fuzzy logic, neural network, gravity compensator is proposed in [1], [2], [3] and [4]. Optimality occurs while achieving stabilization of DIPC around unstable equilibrium point which results into a minimization of quadratic cost function. Solving Riccati equation leads to find the minimum value of cost function. [5] Application of this technique to highly nonlinear system is enacted by Jacobian method explained in [6]. Further applications of double inverted pendulum on a cart like take off stabilization of multi stage rocket are elaborated in [7].

The paper is organized as follows: section 2, discusses the mathematical modeling of double inverted pendulum on a cart. In section 3, control law is implemented using linearization and quadratic regulation approach. In section 4, experimental results are employed and finally some brief conclusions are given in section 5.

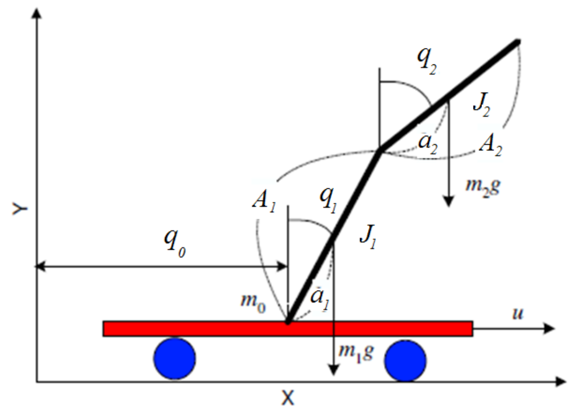

Figure 1: Double inverted pendulum on a cart

II. SYSTEM MODEL

In a DIPC system, two links are connected together on a moving cart as shown in figure 1 [1].

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A. *Mathematical modeling*

A nonlinear system is designed for an expedient result. Lagrangian mechanics yields an ordinary differential equation (actually, a system of coupled differential equations) that describes the evolution of a system in terms of an arbitrary vector of generalized coordinates that completely defines the position of every particle in the system. Euler – Lagrangian equation gives us DIPC's equations of model.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F \quad (1)$$

Where a term "L" is Lagrangian obtained by the difference between total kinetic energy of the system (K) and total potential energy (P) of it. Term "F" represents a generalized forces acting in the direction of generalized coordinates q.

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$$L = Kinetic - Potential \quad (2)$$

Total kinetic energy of the system is given by the sum of individual kinetic energies of cart (K_0), and both pendulums (K_1 and K_2).

$$K_0 = \frac{1}{2} m_1 \dot{q}_0^{\,2} \quad (3)$$

$$K_1 = \frac{1}{2} m_1 \dot{q}_0^{\,2} + \frac{1}{2} (m_1 a_1^{\,2} + J_1) \dot{q}_1^{\,2} + m_1 a_1 \dot{q}_0 \dot{q}_1 \cos q_1 \quad (4)$$

$$K_2 = \frac{1}{2} m_2 \dot{q}_0^{\,2} + \frac{1}{2} (m_2 a_2^{\,2} + J_2) \dot{q}_2^{\,2} + \frac{1}{2} m_2 A_1^{\,2} \dot{q}_1^{\,2} + m_2 A_1 \dot{q}_0 \dot{q}_1 \cos q_1 + m_2 a_2 \dot{q}_0 \dot{q}_2 \cos q_2 + m_2 A_1 a_1 \dot{q}_1 \dot{q}_2 \cos(q_1 - q_2) \quad (5)$$

Similarly potential energy can be represented as P_0, P_1 and P_2 for cart and both pendulums respectively. Initially cart's position is assumed to be at rest. Thus

$$P_0 = 0 \quad (6)$$

And

$$P_1 = m_1 g a_1 \cos q_1 \quad (7)$$

$$P_2 = m_2 g (A_1 \cos q_1 + a_2 \cos q_2) \quad (8)$$

Combining equations (2) to (8), Lagrangian term is obtained as

$$L = \frac{1}{2} (m_0 + m_1 + m_2) \dot{q}_0^{\,2} + \frac{1}{2} (m_1 a_1 + m_2 A_1^{\,2} + J_1) \dot{q}_1^{\,2} + \frac{1}{2} (m_2 a_2^{\,2} + J_2) \dot{q}_2^{\,2} + (m_1 a_1 + m_2 A_1) \dot{q}_0 \dot{q}_1 \cos q_1 + m_2 a_2 \dot{q}_0 \dot{q}_2 \cos q_2 + m_2 A_1 a_2 \dot{q}_1 \dot{q}_2 \cos(q_1 - q_2) - (m_1 a_1 + m_2 A_1)\, g \cos q_1 - m_2 g a_2 \cos q_2 \quad (9)$$

Putting this equation in (1) we get,

$$u = \left(\sum m_i\right) \ddot{q}_0 + (m_1 a_1 + m_2 A_1) \cos q_1 \ddot{q}_1 + m_2 a_2 \cos q_2 \ddot{q}_2 - (m_1 a_1 + m_2 A_1) \sin q_1 \dot{q}_1^{\,2} - m_2 a_2 \sin q_2 \dot{q}_2^{\,2} \quad (10)$$

$$0 = (m_1 a_1 + m_2 A_1) \cos q_1 \ddot{q}_0 + (m_2 a_2^{\,2} + m_2 A_1^{\,2} + J_1) \ddot{q}_1 + m_2 A_1 a_2 \cos(q_1 - q_2) \ddot{q}_2 + m_2 A_1 a_2 \sin(q_1 - q_2) \dot{q}_2^{\,2} - (m_1 a_1 + m_2 A_1) g \sin q_1 \quad (11)$$

$$0 = m_2 a_2 \cos q_2 \ddot{q}_0 + m_2 A_1 a_2 \cos(q_1 - q_2) \ddot{q}_1 + (m_2 a_2^{\,2} + J_2) \ddot{q}_2 - m_2 A_1 a_2 \sin(q_1 - q_2) \dot{q}_1^{\,2} - m_2 g a_2 \sin q_2 \quad (12)$$

As force "F" is acting along only in 'x' direction, here we are considered its component in 'x' direction only i.e. *u*.

B. Compact form

The order of above equation is two. Theses equations can be reduced into simplified and standard format which yields,

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Hu \quad (13)$$

Where "M" is a symmetric mass matrix that expresses the inter relation between the time derivative of the generalized coordinate vector of the system and the kinetic energy of it. Term "C" denotes coriolis and centrifugal terms [8]. Because of non-presence of time derivative of generalized coordinate vector in "G", it can be easily derived from potential energy. Their expanded versions are shown below.

$$M(q) = \begin{pmatrix} z_1 & z_2 \cos q_1 & z_3 \cos q_2 \\ z_2 \cos q_1 & z_4 & z_5 \cos(q_1 - q_2) \\ z_3 \cos q_2 & z_5 \cos(q_1 - q_2) & z_6 \end{pmatrix}$$

$$C(q, \dot{q}) = \begin{pmatrix} 0 & -z_2 \sin q_1 \dot{q}_1 & -z_3 \sin q_2 \dot{q}_2 \\ 0 & 0 & z_5 \sin(q_1 - q_2) \dot{q}_2 \\ 0 & -z_5 \sin(q_1 - q_2) \dot{q}_1 & 0 \end{pmatrix}$$

$$G(q) = \begin{pmatrix} 0 \\ -f_1 \sin q_1 \\ -f_2 \sin q_2 \end{pmatrix}$$

$$H = (1 \quad 0 \quad 0)^T \quad (14)$$

Where

$$z_1 = m_0 + m_1 + m_2$$
$$z_2 = m_1 a_1 + m_2 A_1 = \left(\frac{1}{2} m_1 + m_2\right) A_1$$
$$z_3 = \frac{1}{2} m_2 A_2$$
$$z_4 = m_1 a_1^{\,2} + m_2 A_1^{\,2} + J_1 = \left(\frac{1}{3} m_1 + m_2\right) A_1^{\,2}$$
$$z_5 = m_2 A_1 a_2 = \frac{1}{2} m_2 A_1 A_2$$
$$z_6 = m_2 a_2^{\,2} + J_2 = \frac{1}{3} m_2 A_2^{\,2}$$
$$f_1 = (m_1 a_1 + m_2 A_1) g = \left(\frac{1}{2} m_1 + m_2\right) A_1 g$$
$$f_2 = m_2 a_2 g = \frac{1}{2} m_2 A_2 g \quad (15)$$

Here we have assumed that the centers of mass of the pendulums are in the geometrical center of the links, which are solid rods. Thus, we get: $a_i = A_i / 2$ and $J_i = m_i A_i^2 / 12$ [1].

III. CONTROL DESIGN

A. Linearization

Equation of the system clearly shows that the model belongs to a nonlinear system. Ordinary differential equations can be formed by converting the system into state space model format. State vector of the system is assumed to be as follow.

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$$x = (q \quad \dot{q})^T \tag{16}$$

Linearization is necessary for the application of LQR technique. Nonlinear model of system is given by

$$\ddot{q} = -M^{-1}C\dot{q} - M^{-1}G + M^{-1}Hu \tag{17}$$

From above generalization is easy to deduce the system equation as follow.

$$\dot{x} = \begin{pmatrix} 0 & I \\ 0 & -M^{-1}C \end{pmatrix} x + \begin{pmatrix} 0 \\ -M^{-1}G \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}H \end{pmatrix} u \tag{18}$$

In above equation *I* and *0* are identity and zero matrices of order 3by3.

One of the standard types of nonlinear system can be written as

$$\dot{x} = f(x) + g(x)u \tag{19}$$

Where

$$f(x) = \begin{pmatrix} 0 & I \\ 0 & -M^{-1}C \end{pmatrix} x + \begin{pmatrix} 0 \\ -M^{-1}G \end{pmatrix} \tag{20}$$

$$g(x) = \begin{pmatrix} 0 \\ D^{-1}H \end{pmatrix} \tag{21}$$

Approximated linearization is done with the help of Jacobian matrix. Simply a Jacobian means differentiation which is nothing but linearization. The Jacobian can also be thought of as describing the amount of "stretching", "rotating" or "transforming" that a transformation imposes locally. Approximation is considered about an unstable equilibrium point i.e. at zero state generalized coordinate.

Thus a nonlinear system can be reduced into a standard linear system of the form,

$$\dot{x} = Ax + Bu$$

Where

A and *B* is obtained by Jacobian.

$$A = \frac{\partial f(x)}{\partial x} = \begin{pmatrix} 0 & I \\ -M(0)^{-1}\dfrac{\partial G(0)}{\partial q} & 0 \end{pmatrix} \tag{22}$$

$$B = \frac{\partial g(x)}{\partial x} = \begin{pmatrix} 0 \\ M(0)^{-1}H \end{pmatrix} \tag{23}$$

B. Cost Function

Infinite horizon quadratic continuous time cost function is chosen for the system which is to be minimized.

$$J = \int_0^{\infty} (x^T Q x + u^T R u)\, dt \tag{24}$$

In the infinite-horizon case, however, the matrices Q and R are not only positive-semi definite and positive-definite, respectively, but are also constant. These additional restrictions on Q and R in the infinite-horizon case are enforced to ensure that the cost functional remains positive. Furthermore, in order to ensure that the cost function is bounded, the additional restriction is imposed that the pair (A, B) is controllable.

C. Linear Quadratic Regulator

Optimal control law concerned over here to attain unstable equilibrium position is the solution of the dynamic system where the system dynamics are described by a set of linear differential equations and the cost is described by a quadratic function. This solution is provided by the linear-quadratic regulator (LQR), a feedback controller whose equations are given below.

$$U = -R^{-1}B^T P x = -Kx \tag{25}$$

Where P is the solution of standard Riccati equation which is given by,

$$PA + A^T - PBR^{-1}P + Q = 0 \tag{26}$$

Corresponding value of K i.e. linear feedback control matrix is found used for the simulation.

IV. EXPERIMENTAL RESULTS

Initially the system is assumed to be at rest. Parameters belong to the model are given as follow with their corresponding S. I. units i.e. masses in kg, lengths in meter and inertia in kg-m^2.

$$m_0 = 1.5$$
$$m_1 = 0.5$$
$$m_2 = 0.75$$
$$L_1 = 0.5$$
$$L_1 = 0.5$$
$$L_2 = 0.75$$
$$l_1 = \frac{L_1}{2}$$
$$l_2 = \frac{L_2}{2}$$

$$I_1 = \frac{m_1 L_1 L_1}{12}$$
$$I_2 = \frac{m_2 L_2 L_2}{12} \tag{27}$$

System matrix, input matrix can be obtained from (22), (14), (15) and (27). They are calculated around at upright equilibrium point.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 7.18 & 0.7 & 0 & 0 & 0 \\ 0 & 71.83 & -32.32 & 0 & 0 & 0 \\ 0 & -57.37 & 50.66 & 0 & 0 & 0 \end{bmatrix}$$

$$B = [0 \quad 0 \quad 0 \quad 0.6 \quad -1.46 \quad 0.25]^T$$

While C and D are considered as,

$C = I_6$

where I is an identity matrix of order six and

$D = 0_{6\times1}$

i.e. D is a zero matrix of mentioned dimension.

Effectiveness of LQR problem depends on priority matrices Q and R.

$$Q = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 50 & 0 & 0 & 0 & 0 \\ 0 & 0 & 50 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 \\ 0 & 0 & 0 & 0 & 700 & 0 \\ 0 & 0 & 0 & 0 & 0 & 700 \end{bmatrix}$$

$R = 1$

From A, B, Q and R matrices and with the help of MATLAB environment feedback gain values K are calculated as follow.

$K = [2.24, -528.30, 663.94, 7.07, -19.86, 100.97]$

Assume the initial generalized coordinate vector is

$x_{initial} = [1;\ 0.17; 0.087; 0; 0; 0]$

Following figure clearly shows the stabilization of double inverted pendulum on a cart to an unstable equilibrium position.

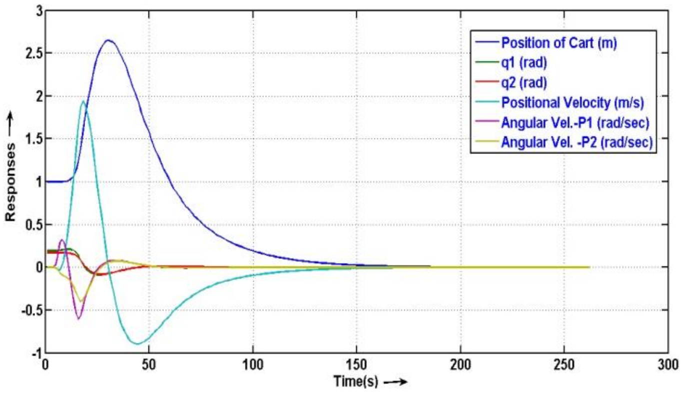

Figure 2: Response of states v/s time

After applying LQR controller, all the states of the system come to an equilibrium point.

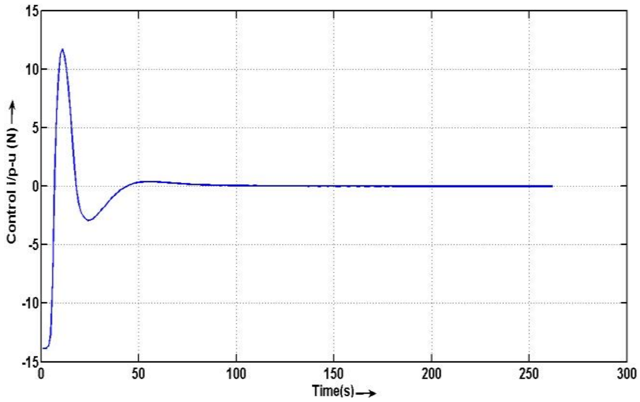

Figure 3: Control input to achieve equilibrium

It shows the adjustment of cart with direction of motion in a negative and positive manner to attain upright unstable equilibrium point.

V. CONCLUSION

A highly nonlinear system can stabilize by linearization of it around an equilibrium point. It forms exact linear system for the application of Linear Quadratic Regulator problem. Balancing of double inverted pendulum on a cart is successfully carried out by linearization technique.

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