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A Contribution to the Computer Assisted, Kinematics, Kinetostatics, and Dynamic Analysis of the Inline and Offset types of the Crank-and-Connecting Rod Mechanism

Tonye K. Jack

Abstract*— In this paper a simple spreadsheet program method of calculations for the complete kinematics, kinetostatics, and dynamic analysis of the inline and offset types of the crank-and-connecting-rod mechanism is presented. Being a single degree-of-freedom mechanism as defined by its crank angle, the program can be used to answer "what-if" scenario questions through tables and graphical plots to evaluate variations of key motion and loading parameters with changes in the crank angle. The program also allows for the conduct of parameter studies in selecting optimum crank-and-connecting-rod linkage dimensions and speeds. Extreme positions are accounted for in the inline model using the Ching-U and Price model equations. An equation derived for the offset model, estimates and predicts relative crank angle position, and relative extremum maximum velocity, to within 93-to-96 % of actual absolute extremum maximum piston velocity guided by applications of the extreme value principle.

Index Terms*—Crank-and-Connecting-Rod Mechanism, Inline Slider-Crank, Inverse Dynamics, Kinematics of Mechanisms, Kinetostatics, Mechanisms, Mechanism Synthesis, Offset Slider-Crank, Slider-Crank Mechanism.

I. INTRODUCTION

Computer programs (visual and non-visual Graphics packages) for integrated Mechanisms analysis have been reported in the literature; the programs are mostly written in FORTRAN and the BASIC languages and used for kinematics, kinetostatic and Dynamics analysis of two- and three-dimensional rigid link mechanisms [1]-[9]. Kinetostatic analysis or inverse dynamics, allows for the computation of the bearing reaction forces, and the required shaft input torque at a particular instant position of the crank angle [9]. Kaplan and Pollick [10], presented a tabulating scheme for kinematics analysis of a four-bar linkage. The program method of analytical solution presented in this article follows the tabulation approach with the added advantage of the clear input/output formatting and presentation, provided by the Microsoft Excel ™ spreadsheet environment. Bearing Friction and the concept of friction circle is not accounted for in this program model.

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* Correspondence Author
Tonye K. Jack teaches Mechanical Engineering classes in Port Harcourt, Rivers State, Nigeria.

II. REQUIRED KINEMATICS EQUATIONS

The basic model of an inline crank-and-connecting rod mechanism is shown in Fig. (1). Shown in Fig. (2), is the offset crank model type of mechanism.

The mathematical relations for the analysis of these mechanism types are derived from these basic geometries. Some of the derivations of these equations are available in the open literature. Complete step-by-step analytical derivations of the kinematics, kinetostatic and dynamic analysis equations of the inline model are given in [11], [7]. Khurmi [5], assuming negative connecting rod angle, derived mathematical relationships for the displacement, velocity, and acceleration for the offset model. Freudenstein and Sandor [6] presented a set of equations for the offset model, which adopted a positive connecting rod angle. Doughty [8], using a different approach with generalized coordinates and velocity coefficients also derived mathematical relations for the offset type mechanism. For the program discussed in this article, the offset type model equations of [5] are applied.

A. Design Factors

The design factor, f_d, defines the geometric link between the crankshaft and connecting rod, and is obtained by the trigonometric analysis of the crank-and-connecting rod linkage mechanism triangular geometry. This is obtained as (1):

$$f_d = LSin\beta = rSin\theta \quad (1)$$

Equation (1) defines the important design parameter ratio of connecting rod length, L, to the crank radius, r, and obtained as (2): $\frac{L}{r}$ Ratio:

$$\frac{L}{r} = \frac{Sin\theta}{Sin\beta} \quad (2)$$

The importance of the (L/r) ratio for the inline mechanism model is its usefulness in defining the crank motion type in terms of the reciprocal value, i.e. (r/L) ratio:
For $(r/L) < 1$, crank motion is rotating; for $(r/L) > 1$ crank motion is oscillatory [6].

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B. Stroke Length

Inline: $S_t = 2r$ (2a)

For the offset stroke length, [6], [7] provide the following:

Offset: $S_t = \sqrt{(L+r)^2 + e^2} - \sqrt{(L-r)^2 - e^2}$ (2b)

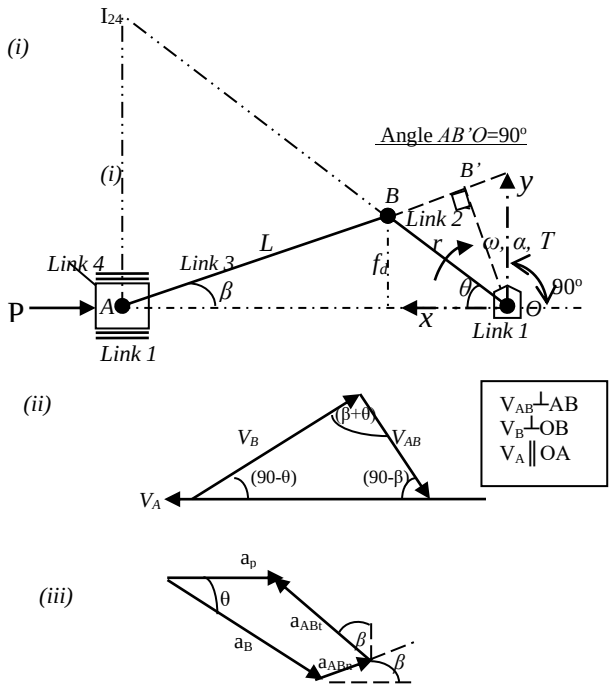

Fig. 1: Basic Inline Model Crank-and-Connecting-Rod Mechanism *(i)*; inset: velocity triangle *(ii)* and acceleration

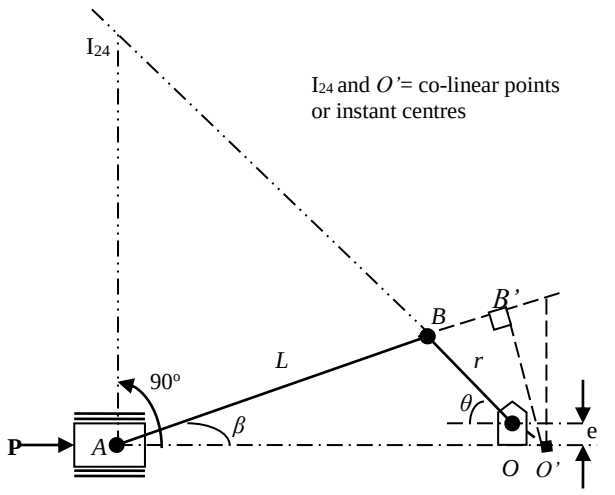

Fig. (2): Basic Offset Model Crank-and-connecting rod

C. Connecting Rod Angle

In Fig. (1), the connecting rod angle, β, for the inline model is obtained from (1) as:

$$\beta = Sin^{-1}\left[\left(r/L\right)Sin\theta\right] \quad (3)$$

The connecting rod angle for the offset type model of fig. (2) is as in (3a):

$$\beta = Sin^{-1}\left[\frac{e - r\sin\theta}{L}\right] \quad (3a)$$

Wilson, Sadler and Michels [7], set the limiting conditions for the (L/r) ratio in design solution and analysis, applicable to the inline model as:

For *Exact* condition: $\frac{L}{r} < 3$;

And, for *Approximate* condition: $\frac{L}{r} \geq 3$.

Others report different limiting conditions [12]-[13]. Khurmi, [5] makes no such distinction in the application of the kinematics relationships for the offset model.

D. Piston Displacement: Inline Model

The conversion of the rotary angular displacement of the crankshaft, to the translational linear displacement of the piston between the bottom dead centre and the top dead centre is defined by the piston displacement relations, (4) or (4a) for the inline model. This is based on a position analysis of the crank-and-connecting rod mechanism when viewed from the full extended position. Depending on the (L/r) ratio, the following relations apply:

For $\frac{L}{r} < 3:-$

$$s = L[1 - Cos\beta] + r[1 - Cos\theta] = r[1 - Cos\theta] + L\left\{1 - \left[1 - \left(r/L\right)^2 Sin^2\theta\right]^{\frac{1}{2}}\right\} \quad (4)$$

For $\frac{L}{r} \geq 3:-$

$$s = r\left\{[1 - Cos\theta] + \left(1/4\right)\left[\left(r/L\right)\left(1 - Cos2\theta\right)\right]\right\} \quad (4a)$$

Equation (4a) is an approximate relationship and obtained from the Taylor series expansion of the last radical term in enveloped bracket of (4) [11].

E. Piston Displacement: Offset Model

Based on the formulations of [5], the offset model piston displacement is obtained from a solution of the quadratic, (4b):

$$s^2 + k_1 s + k_2 \quad (4b)$$

Where,

$$k_1 = -2r\cos\theta \quad (4c)$$

And

$$k_2 = r^2 - L^2 + e^2 - 2er\sin\theta \quad (4d)$$

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F. Piston Velocity: Inline Model

The corresponding piston velocity obtained from the first derivative of piston displacement with respect to time is:

For $\frac{L}{r} < 3:-$

$$v = \frac{ds}{dt} = \omega r Sin\theta \left[1 + \left(\frac{r}{L}\right) \frac{Cos\theta}{\sqrt{1 - \left(\frac{r}{L}\right)^2 Sin^2\theta}} \right] \quad (5)$$

For $\frac{L}{r} \geq 3:-$

$$v = \omega r Sin\theta \left[1 + \left(r/L \right) Cos\theta \right] \quad (5a)$$

Again, the simplified (5a) is obtained from the Taylor series expansion of the binomial form of the denominator of the square root term in (5).

G. Piston Velocity: Offset Model

Based on [5],

$$v = \frac{r\omega Sin(\beta - \theta)}{Cos\beta} \quad (5b)$$

H. Crank Pin Velocity and Relative Velocity of Piston to Crank Pin

Analysis of the inset velocity triangle in Fig. 1 *(ii)*, by the sine rule for triangles based on the direction of motion results in the relationship:

$$\frac{V_A}{Sin(\beta + \theta)} = \frac{V_B}{Sin(90 - \beta)} = \frac{V_{AB}}{Sin(90 - \theta)} \quad (6)$$

The crank pin velocity, V_B, is also obtainable from (6a):

$$V_B = r\omega \quad (6a)$$

I. Angular Velocity of the Connecting Rod: Inline Model

$$\omega_{AB} = \frac{V_{AB}}{L} \quad (7)$$

J. Angular Velocity of the Connecting Rod: Offset Model

Again, based on derivation of [5],

$$\omega_{AB} = \frac{-r\omega Cos\theta}{LCos\beta} \quad (7a)$$

K. Piston Acceleration: Inline Model

The piston acceleration is obtained from the second derivative of the displacement with respect to time.

For $\frac{L}{r} < 3:-$

$$a_p = \frac{d^2s}{dt^2} = \omega r^2 \left\{ Cos\theta + \frac{\left(r/L\right)^3 Sin^4\theta + \left(r/L\right) Cos2\theta}{\left[1 - \left(r/L\right)^2 Sin^2\theta\right]^{\frac{3}{2}}} \right\} \quad (8)$$

For $\frac{L}{r} \geq 3:-$

$$a_p = \omega r^2 \left[Cos\theta + \left(r/L\right) Cos2\theta \right] \quad (8a)$$

L. Piston Acceleration: Offset Model

This is given by [5] as in (8b),

$$a_p = \frac{r\alpha Sin(\beta - \theta) - r\omega^2 Cos(\beta - \theta) - L\omega_{AB}^2}{Cos\beta} \quad (8b)$$

M. Crank Pin Acceleration:

The crank pin acceleration is made up of normal and tangential components.

Note that, in the Fig. (1) and Fig. (2), the crank – Link 2 – rotates with an angular velocity, ω, and an angular acceleration, α. In many instances, the crank rotates at a uniform angular velocity [12].

For constant angular velocity of the crank, the normal acceleration component is:

$$a_B = a_{Bn} = \omega^2 r = \left(\frac{\pi N}{30}\right)^2 r = \frac{V_B^2}{r} \quad (9)$$

If the crank rotates with a constant angular velocity, ω, the tangential component of crank pin acceleration, a_{Bt} =0.

If the crank rotates with angular velocity, ω, and an angular acceleration, α, the tangential component of crank pin acceleration is given as in (9a):

$$a_{Bt} = r\alpha \quad (9a)$$

The resultant crank pin acceleration is then the vector addition:

$$\vec{a}_B = ia_{Bn} + ja_{Bt} \quad (9b)$$

N. Relative Normal Acceleration of Piston to Crank Pin:

$$a_{ABn} = \omega_{AB}^2 L \quad (10)$$

O. Relative Tangential Acceleration of Piston to Crank Pin

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$$a_{ABt} = \frac{mCos\theta + nCos\beta - a_p}{Sin\beta} \quad (11)$$

Where,

$m=a_{Bn}$, $n=a_{ABn}$ (11a)

Equation (11) can be deduced from the inset acceleration polygon of Fig. (1) *(iii)*.

P. Angular Acceleration of Connecting Rod: Inline Model

$$\alpha_{AB} = \frac{a_{ABt}}{L} \quad (12)$$

Q. Angular Acceleration of Connecting Rod: Offset Model

By [5] formulation,

$$\alpha_{AB} = \frac{r(\alpha Cos\theta - \omega^2 Sin\theta) - L\omega_{AB}^2 Sin\beta}{LCos\beta} \quad (12a)$$

III. EXTREME POSITIONS: ANALYSING FOR CRANK POSITIONS AT MAXIMUM PISTON VELOCITY

A. Crank Angle at Maximum Piston Velocity: Inline Model

Ching-U and Price [14] applied Lin's successive division method to the solution of a characteristic cubic equation of the position of maximum piston velocity, derived from the exact relationship for piston acceleration, (8), based on the fact that, at the point of maximum velocity, the acceleration will be zero. This is in line with the maxima-minima theory. Equation (13), obtained from the [14] solution can be used to compute crank angle position at maximum piston velocity.

For $\frac{L}{r} < 3:-$

$$Cos\theta = \left[\left(L/r\right)^2 + 3\right]^{-\frac{1}{2}} \quad (13)$$

Ching-U and Price [14], also give (13a) as the crank angle position at maximum piston velocity from a solution of the approximate (8a).

For $\frac{L}{r} \geq 3:-$

$$Cos\theta = \left(-\frac{L}{4r} + \left[\left(\frac{L}{4r}\right)^2 + 0.5\right]^{\frac{1}{2}}\right) \quad (13a)$$

B. Estimating and Predicting Crank Angle at Maximum Piston Velocity for the Offset Model

At the point of maximum velocity, the acceleration is zero. Equation (8b) then reduces to, (13b):

$$r\alpha Sin(\beta - \theta) - r\omega^2 Cos(\beta - \theta) - L\omega_{AB}^2 = 0 \quad (13b)$$

A close observation of the offset model (5b) and (7a) relations, show that, maximum piston velocity, V_{pmax}, and maximum angular velocity of the connecting rod, $\omega_{AB\text{-}max}$, will occur in the first and fourth quadrant of the connecting rod angle, at $0 < \beta < \frac{\pi}{2}$, and $\frac{3\pi}{2} < \beta < 2\pi$, with infinite point at $\pi/2$ and $3\pi/2$, *since, (Cos β= Cos $\pi/2$ = Cos$3\pi/2$= 0); and thus, with the finite maximum likely occurring just below, $\beta=\pi/2$,* i.e., *$(\pi/2)^-$, and/or, just over $\beta=3\pi/2$,* i.e., *$(3\pi/2)^+$.*

Plots of offset type piston velocity versus crank angle indicate the curves are not symmetric, indicating several points of maximum, minimum, and inflection curve points. This differs from the inline model, wherein the curves are antisymmetric, about the crank angle, π, and with clear extreme values [7]. This can also be observed in the numerical examples in this article.

Setting the value of $\beta = \pi/2=90^o$, (13b) can be re-written as in (13c):

$$r\alpha Sin(90 - \theta) - r\omega^2 Cos(90 - \theta) - L\omega_{AB}^2 \equiv r\alpha Cos\theta - r\omega^2 Sin\theta - L\omega_{AB}^2 = 0 \quad (13c)$$

Or

$$r\alpha Cos\theta - r\omega^2 Sin\theta = L\omega_{AB}^2 \quad (13d)$$

Next is to define ω_{AB} at maximum condition. An estimate of the point of maximum velocity can be obtained by the maxima-minima principle of: given a function, $\omega_{AB}(\theta)$, in this case the angular velocity of the connecting rod, which has a local maximum point, a local minimum point, and an inflection, by the maxima-minima principal, a stationary point, θ, can be obtained by equating the solution of : *(1.)* first derivative to zero, i.e. , $\frac{d\omega_{AB}}{d\theta} = 0$; *(2.)* also, another necessary condition is, the second derivative is less than zero, i.e., $\frac{d^2\omega_{AB}}{d\theta^2} < 0$, [15].

Applying the first condition to (13d), a good estimate of the offset model type crank angle at maximum velocity is predictable within a 4 %-to-7 % error margin with the following relation:

$$\tan\theta = \frac{-\omega^2}{\alpha} \quad (13e)$$

Or

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$$\theta = \tan^{-1}\left(\frac{-\omega^2}{\alpha}\right) \qquad (13f)$$

(See appendix)

Since (13f) is a relative extremum crank angle at maximum velocity, it can be reasoned that actual crank angle at maximum velocity is of the form of, (13g) or (13h):

$$\theta = \left\{\left[\tan^{-1}\left(\frac{-\omega^2}{\alpha}\right)\right] + 360\right\} + q \qquad (13g)$$

Or

$$\theta = \left\{\left[\tan^{-1}\left(\frac{-\omega^2}{\alpha}\right)\right] + 360\right\}\eta \qquad (13h)$$

Equation (13h) is easier to apply in estimating crank angle position. By test inspection, the factor, η, is empirically within the limit of 1.04 – 1.07. A figure of 1.055 is good. Alternatively, a direct approach can be applied by multiplying the relative extremum maximum velocity obtained by applying (13f), by the factor, η, to obtain close to actual maximum velocity. Using the tables and plots of the parameter studies discussed later, further simplifies the computation.

By rewriting (13g) in the form of (13i):

$$\theta = \left\{\left[\tan^{-1}\left(\frac{-\omega^2}{\alpha}\right)\right] + 360\right\} + q \equiv \theta_1 + q_1\theta_1 \qquad (13i)$$

The factor, q_1, is estimated to be within the range, 0.04-0.07. Note that factors, q, and, η, are not adjusted for in the program. This is left to the judgement of the program user.

The method applied to estimate the crank angle at maximum velocity for the offset model agrees with the Freudenstein extreme value theorem, and the follow-up inversion theorem for mechanisms, in terms of determining angle phases of a linkage in relation to the extremes of velocity ratio [12]. Shigley [12], further reports that, Freudenstein advices that in the evaluation of a linkage to fulfill a motion position coordinate, the linkage be investigated for maximum, minimum or a point of inflection.

In the case for the offset model, the predicted crank angle is an estimated position value to give an idea of the location of crank angle position at the extreme maximum velocity. Due to the nonsymmetric nature of the offset model as seen from the velocity plot, with increasing and decreasing, upward concavity, and decreasing and increasing, downward concavity, clear extremes will need to be by additional trial, guided from the estimated. This is an important consideration of the *Extreme Value Property* as defined by [16]-[17]. By the Extreme Value Property, "a continuous function within a given interval will attain absolute extrema at the end points or at a critical point within the interval" [16]-[17]. Note that absolute extrema is the collective term for the absolute maxima and minima [16].

The Extreme Value Property thus, provides answers to two questions: (1) what happens when a continuous function cannot be easily maximized or minimized at the end points? (2) What should be done if an absolute extrema does really exist in such cases of unbounded intervals?

Hoffman, Bradley and Rosen [16, 17], provide a useful guide with this Quote:

"When an absolute extremum does exist and a function is continuous on an interval, the absolute extremum will still occur at a relative extremum or end point contained in the interval." The relative extrema are critical points contained in the interval.

Hoffman, Bradley and Rosen [16, 17], further advice that, to find the absolute extrema of a continuous function on an interval, an evaluation of all the critical points and end points that are contained in the interval is required. Hoffman, Bradley and Rosen [16, 17], however, caution that, before any final conclusions are drawn, it is important to find out if the function actually has relative extreme on the interval. Applying the first derivative to determine where the function is increasing and where it is decreasing, together with a plot sketch is a useful step [16, 17].

Equation (13f) thus, predicts the relative extremum, and allows for the absolute extremum to be estimated by way of (13g) or (13h).

IV. REQUIRED KINETIC EQUATIONS – TORQUE AND POINT MASSES

A. *Static Input Torque of Crankshaft*

$$T = -\Pr Sin\theta\left\{1 + \frac{\left(r/L\right)Cos\theta}{\left[1-\left(r/L\right)^2 Sin^2\theta\right]^{\frac{1}{2}}}\right\} \qquad (14)$$

B. *Dynamic Torque of Crankshaft*

$$T = -\left[I_{G2}\alpha + F_{32x}r\sin\theta + F_{32y}r\cos\theta\right] \qquad (15)$$

Note that when the mechanism operates at a constant angular velocity, the first product term in (15) is zero, since, $\alpha=0$.

C. *Piston Mass and Dynamic Equivalent Mass Modeling*

Letting the connecting rod mass = m_3, and assuming that it is a mass-less rigid link being held in its equilibrium position, by two point masses, m_A and m_B at its ends A and B respectively, with the centre of mass of the connecting rod at G_3, - see Fig. (3) - the following three relations, (16), (16a), and (16b) can be obtained from equilibrium consideration of the Link 3, [18], [7], [11]:

$$m_B = \left(\frac{L_A}{L}\right)m_3 \qquad (16)$$

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$$m_A = \left(\frac{L_B}{L}\right) m_3 \tag{16a}$$

$$m_A L_A^2 + m_B L_B^2 = I_{G3} \tag{16b}$$

Where, L_A and L_B are the distances of the gudgeon pin, A, and crank pin B, from the centre of mass of connecting rod, G_3.

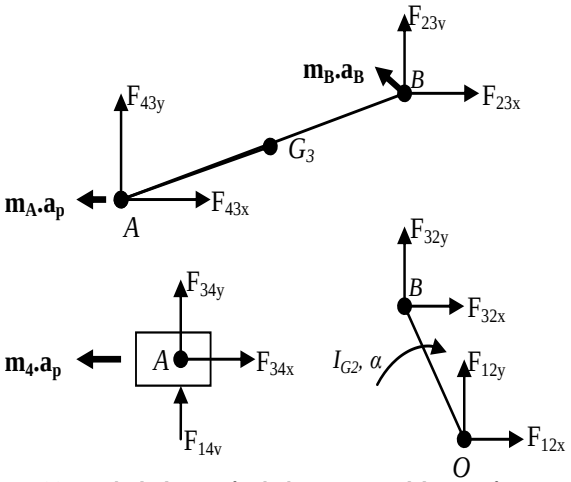

Fig. (3): Free body diagram for the kinetostatic and dynamic force analysis [11], [7].

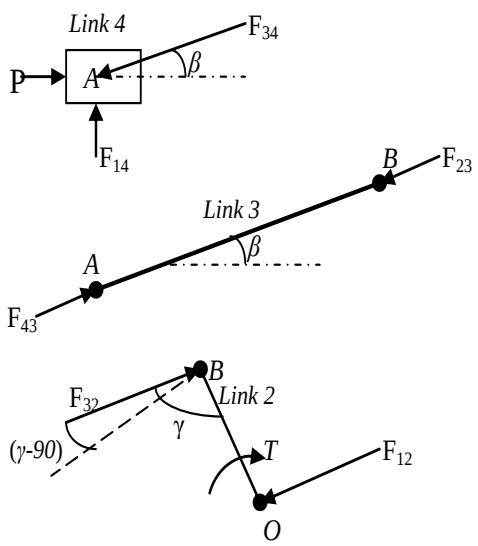

Fig. (4): Free body diagram for the static force analysis [11], [7].

V. REQUIRED KINETIC EQUATIONS – FORCE ANALYSIS

In Fig. (3) and Fig. (4), the free-body diagrams for the dynamic, kinetostatic, and static force analyses of the crank-and-connecting-rod mechanism are shown. Values of the forces acting at critical linkage and bearing points can be calculated from the following equations below. The derivations can be obtained from [11], [7].

Forces acting on Connecting rod and Crank for static equilibrium,

$$F_{32} = F_{34} = \frac{P}{\cos\beta} \tag{17}$$

A. *Main Crankshaft Bearing*

X-component of Bearing Reaction Force at the Main Crankshaft bearing, $F_{12x} = (-F_{32x})$,

$$F_{12x} = -F_{32x} = (m_4 + m_A)a_p + m_B a_{Bx} \tag{18}$$

Where,

$$a_{Bx} = (-r\cos\theta)\omega^2 \tag{18a}$$

Y-component of Bearing Reaction Force at the Main Crankshaft bearing, $F_{12y} = (-F_{32y})$,

$$F_{12y} = -F_{32y} = [(m_4 + m_A)a_p]\tan\beta - m_B a_{By} \tag{18b}$$

Where,

$$a_{By} = (-r\sin\theta)\omega^2 \tag{18c}$$

Resultant Bearing Reaction Force at the Main Crankshaft bearing, F_{12},

$$\vec{F}_{12} = iF_{12x} + jF_{12y} \tag{18d}$$

B. *Crank Pin Bearing*

X-component, Bearing Reaction Force at the Crank Pin, F_{23x},

$$F_{23x} = (m_4 + m_A)a_p + m_B a_{Bx} \tag{19}$$

Y-component, Bearing Reaction Force at the Crank Pin, F_{23y},

$$F_{23y} = [(m_4 + m_A)a_p]\tan\beta - m_B a_{By} \tag{19a}$$

Total Bearing Reaction Force at the Crank Pin, F_{23},

$$\vec{F}_{23} = iF_{23x} + jF_{23y} \tag{19b}$$

C. *Gudgeon Pin Bearing*

Horizontal -*X*-component Bearing Reaction Force at the Gudgeon Pin, F_{43x},

$$F_{43x} = -m_4 a_p \tag{20}$$

Vertical -*Y*-component Bearing Reaction Force at the Gudgeon Pin, F_{43y},

$$F_{43y} = -[(m_4 + m_A)a_p]\tan\beta \tag{20a}$$

Resultant Bearing Reaction Force at the Gudgeon Pin, F_{43},

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$$\vec{F}_{43} = iF_{43x} + jF_{43y}$$
(20b)

D. Cylinder Frame

Reaction Force of Cylinder on Piston, F_{14y},

$$F_{14y} = F_{43y} = -[(m_4 + m_A)a_p]\tan\beta \qquad (21)$$

Crank-and-connecting rod model design with crankshaft offset from line of axis of the piston, allows for the side thrust on the cylinder wall to be minimized during the firing stroke. This is known as the 'desaxe arrangement [13].

VI. DYNAMIC FORCE BALANCING

A. Balance counterweight mass correction- mcrc

The inertia forces can cause reciprocating machine unbalance. A counterweight mounted on the crank, as in Fig. (5), is used to reduce the inertia forces produced at the crank, and thus, force balance the machine. With a counterweight mass, m_c, at a radial distance, r_c from the main bearing, and at an angle ($\theta+180$), [7] suggests balance mass correction amounts, should typically, be in the range:

$$\frac{1}{2}(m_4 + m_A)r \le m_c r_c \le \frac{2}{3}(m_4 + m_A)r\,. \qquad (22)$$

The Total shaking force, F_s, is then given by the form of the F_s, force vector relation of (23):

$$\vec{F}_s = i(F_{12x}) - i(F_{Cx}) - j(F_{Cy}) \qquad (23)$$

At the guide frame, $m_B = 0$, and the relation in (18) for F_{12x} in (23) is reduced to:

$$F_{12x} = (m_4 + m_A)a_p \qquad (24)$$

This being the critical shaking force to be eliminated.

In (24), the inertia force is made up of a primary inertia force, that rotates at the crank speed, ω, and a secondary inertia force, which rotates at twice the speed of the crankshaft [18], [7]. Complete balance is however, not achieved because of the -*y*-component term. The –*x*-component terms as can be seen by (23), is reduced [7].

In (23), F_{Cx} and F_{Cy} are respectively, the -*x*- and -*y*-components of the centrifugal inertia force, $F_C = m_c\omega^2 r_c$ defined by the relations given in (25) and (25a):

$$F_{Cx} = m_c\omega^2 r_c(Cos\theta) \qquad (25)$$

$$F_{Cy} = m_c\omega^2 r_c(Sin\theta) \qquad (25a)$$

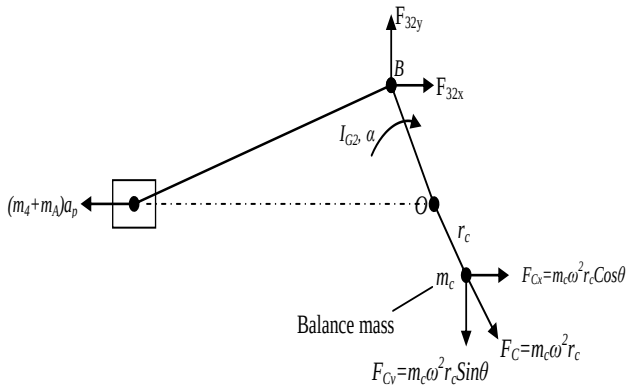

Fig. (5): Shaking Force balance mass with counterweight [18], [7]

VII. PARAMETER STUDIES [7]

The table and plots of the ratio of Piston velocity – to – Crank pin velocity, $z = \left(V_p / V_B\right)$ versus Crank angle for a particular $\left(L/r\right)$ ratio, is a useful aid for conducting parameters studies in calculating piston velocity at any crank position, simply by multiplying the product, ωr, by the ratio, $z = \left(V_p / V_B\right) = \left(V_p / r\omega\right)$, since $V_B = \omega r$.

Again, knowing the $z = \left(V_p / V_B\right)$ ratios for a range of crank angle positions, ($\theta_{p1} \le \theta \le \theta_{p2}$), say, [$z_1 \le z \le z_2$)], mechanism linkage dimensions to fulfill the requirements of certain limiting motion conditions, say, ($V_{p1} \le V_p \le V_{p2}$), can be obtained by the following crank radius, r_1, relation:

$$r_1 = \frac{V_{p1}}{\omega\left(V_p / V_B\right)_1} = \frac{V_{p1}}{z_1\omega} \qquad (26)$$

Note that for this case, V_{p1} is minimum piston velocity, and V_{p2} is maximum piston velocity.

Having obtained the crank radius, r_1, as in (26), next check that the maximum velocity, V_{p2}, falls within the limiting range, ($V_{p1} \le V_p \le V_{p2}$), by the simple calculation:

$$\frac{v}{r_1\omega} = \left(\frac{V_p}{V_B}\right)_2 = z_2$$

Then,

$$v = r_1\omega\left(\frac{V_p}{V_B}\right)_2 = r_1\omega(z_2) \qquad (27)$$

If the value of, v, obtained in (27) falls within the limiting velocity range, ($V_{p1} \le V_p \le V_{p2}$), then it implies that, the speed conditions are approximately satisfied within the selected crank angle range, ($\theta_{p1} \le \theta \le \theta_{p2}$), and, thus, a tentative solution for connecting rod length can be obtained from:

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$$L = r\left(L/r\right)$$

A more accurate computation of the piston velocity then follows using the *(L/r)* ratio, and the actual crank angles, $\theta_{p1} \leq \theta \leq \theta_{p2}$).

Parameters Studies are particularly essential in locating the point of crank angle absolute extremum for maximum velocity consideration when the method applied in this article locates the crank angle at relative extremum and thus, the relative extremum maximum velocity for the offset model.

The $\left(V_p / V_B\right)$ ratios versus crank angle plots have been done in the program with crank angle interval spacing of 15°. For a better optimized study, smaller interval spacing can be adopted. This will also reduce or eliminate the tests for locating the absolute extremum of crank angle for maximum velocity.

VIII. PROGRAM DRIVERS

The program is a single screen, multifunctional, and can be used for crank-and-connecting-rod of the inline and offset types. Formulae and Microsoft Excel macro functions are directly programmed, as in-cell value generators, once defined inputs are provided. The following macro formula distinguishes between an inline and offset type:

IF(B3=0,"INLINETYPECRANK-AND-CONNECTING-ROD MECHANISM-offset=",IF(B3>0,"OFFSET TYPE CRANK-AND-CONNECTING-RODMECHANISM-offset(mm)"))

Where, cell "B3" is the offset or eccentricity, for which inserting, 0, i.e., zero, in cell B3, activates program for inline calculations. The reverse also olds for activating program for offset model calculations, i.e., B3>0.

Another program driver is the crank speed, when operating at a constant crank speed, and when operating with angular velocity and angular acceleration. The macro formula for this case is:

IF(B2=1,"INPUT PARAMETERS when Operating at Constant Crank Speed",IF(B2=2,"INPUT PARAMETERS-Operating with angular vel. and ang. acceleration"))

A most important program driver is the (L/r) ratio. As observed from close inspection of the piston displacement macro formula below, the cell "B16" conditionality is, B16<3 which represents the exact relations, and B16>=3, which is for the approximate. That is for the inline model. No such distinction is made for the offset type, for which the program driver remains, the offset, "B3>0", and "B2=2", and using the [5] offset model piston displacement relation, the absolute value of which is programmed in as the Microsoft Excel function type: ABS (number)

IF(B3=0,IF(B16<3,((B5*(1-COS((PI()/180)*B20)))+(B4*(1-COS((PI()/180)*B8)))),IF(B16>=3,((B4)*((1-COS((PI()/180)*B8))+((0.25)*((B17)*(1-COS((PI()/180)*2*B8))))))),IF(B3>0,ABS((0.5*((2*B4*(COS((PI()/180)*B8)))+((((-2*B4*(COS((PI()/180)*B8)))^2)-(4*((B4^2)-(B5^2)+(B3^2)-(2*B3*B4*(SIN((PI()/180)*B8))))))^0.5))))))

IX. NUMERICAL EXAMPLES

A. Numerical Example1: Inline Model – Exact relationship with (L/r) <3

Given the following data for a crank and connecting rod arrangement: connecting rod length, L=100 mm, crank radius = 40 mm, crank angle, θ=45°, constant crank speed, N= 1200 rpm, Gas Force = 50 N, crank angular acceleration, α = 0 rad/s^2. Piston mass = 0.45 kg, Moment of Inertia of crank-end, I_{G2} = 0.125 N m^2 and the connecting rod mass = 0.25 kg. The ratio of length of the crank pin position from G_3, to the full length of the connecting rod = 0.55. Dynamic balancing of this mechanism will require adding a correction weight of $m_c r_c = 0.6\,(m_4+m_A)r$. Conduct a detailed analysis to determine the displacement, velocity and acceleration of the piston, crank pin velocity and acceleration, and the angular velocity and acceleration of the connecting rod for this constant crank speed and crank position? What are the bearing reaction forces at key joints' sections? Determine the static torque and dynamic torque at θ = 45°. Create plots to analyse variations of motion and kinetic parameters at selected varying crank positions. Neglect Friction.

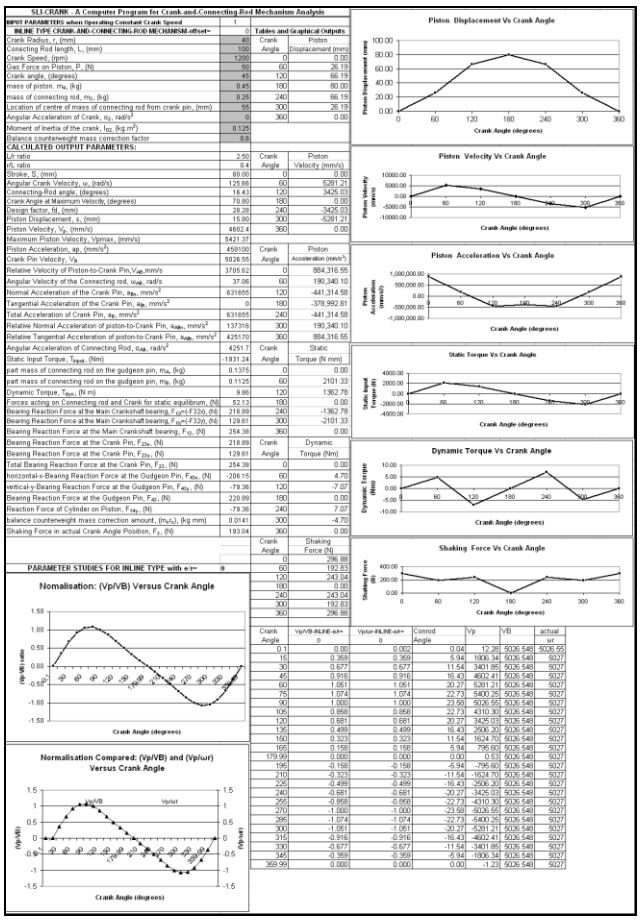

Fig. (6): Excel Program Sheet for Numerical Example 1

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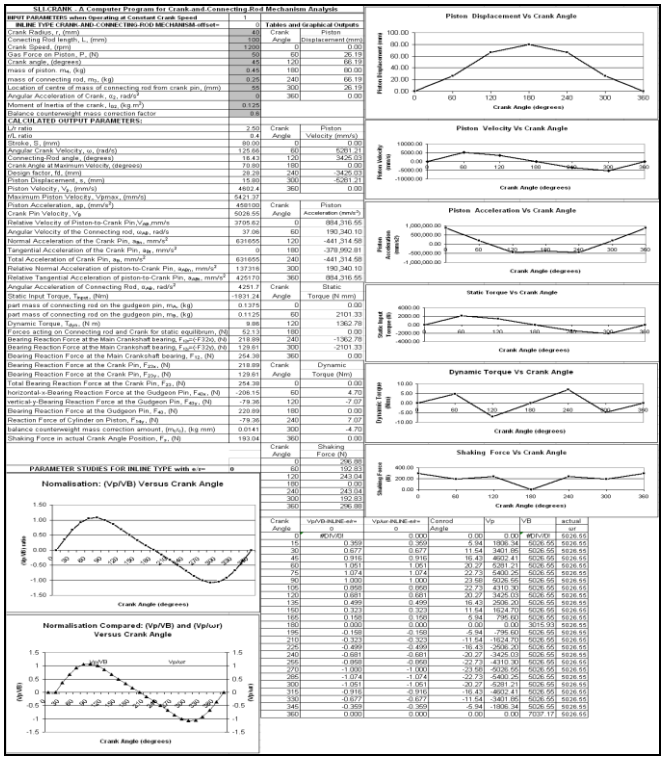

Fig. (7): Numerical Example 1 result with structural error

B. Numerical Example 2: Inline Model – Approximate relationship with (L/r) ≥3

Conduct a detailed motion and force analysis of an inline crank-and-rod-mechanism given the following data for a crank and connecting rod arrangement: connecting rod length, L=100 mm, crank radius = 31.25 mm, crank angle, θ=45°, constant crank speed, N= 850 rpm, Gas Force = 53.5 N, crank angular acceleration, $\alpha = 0$ rad/s^2. Piston mass = 0.35 kg, Moment of Inertia of crank-end, I_{G2} = 0.125 N m^2 and the connecting rod mass = 0.16 kg, the ratio of length of the crank pin position from G_3, to the full length of the connecting rod = 0.55. Dynamic balancing of this mechanism will require adding a correction weight of $m_c r_c = 0.6\,(m_4+m_A)r$. Neglect Friction.

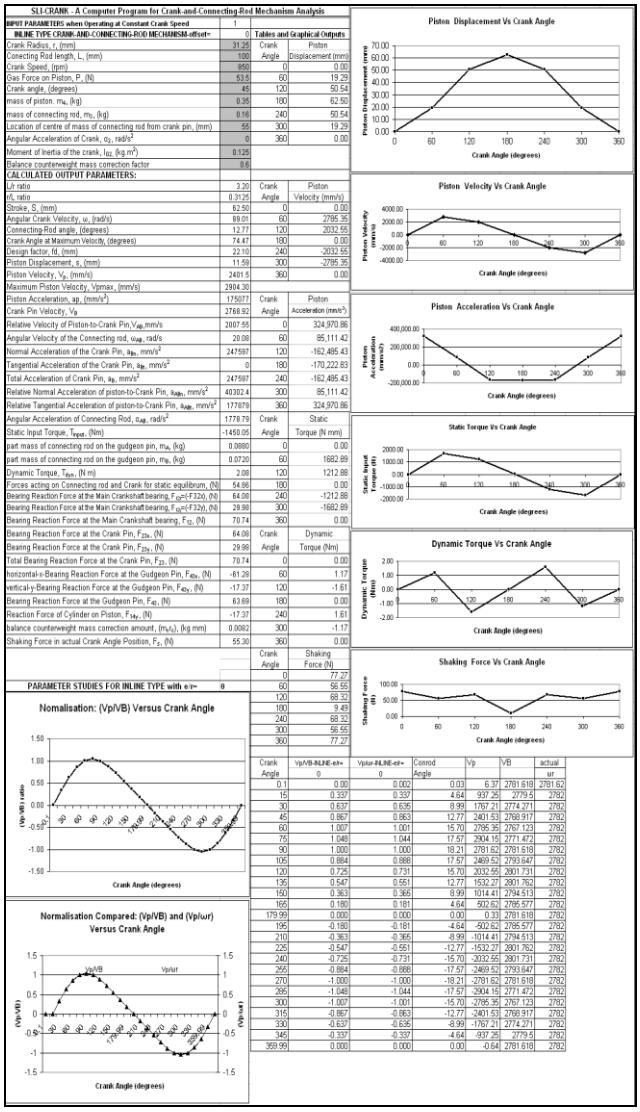

Fig. (8): Excel Program Sheet for Numerical Example 2

C. Numerical Example 3: Offset Model – Approximate relationship with (L/r)≥3

Conduct a detailed motion and force analysis of the following offset type crank-and-connecting rod arrangement: Connecting rod length, L=750 mm, crank radius = 200 mm, crank angle, θ=30°, crank speed, N= 190.989 rpm, angular acceleration, $\alpha = 10$ rad/s^2

(The above offset model geometry and motion data were taken from [5], Example 25.2, pp. 1016, as part of the program validation checks)

Gas Force = 53.5 N, Piston mass = 0.35 kg, Moment of Inertia of crank-end, I_{G2} = 0.125 N m^2 and the connecting rod mass = 0.16 kg. The ratio of length of the crank pin position from G_3, to the full length of the connecting rod = 0.55. Dynamic balancing of this mechanism will require adding a correction weight of $m_c r_c = 0.6\,(m_4+m_A)r$. Neglect Friction.

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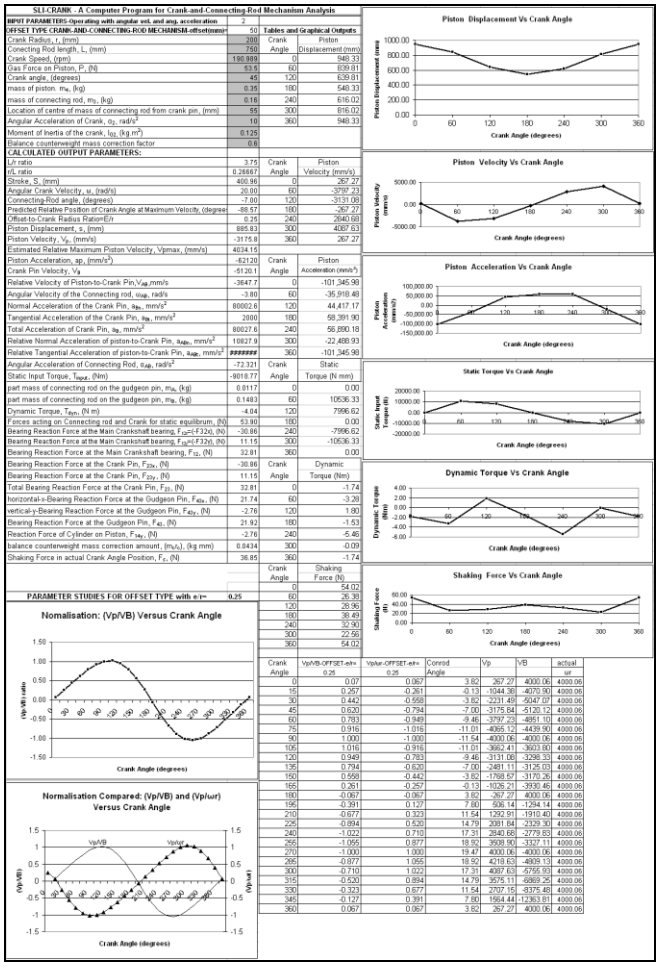

Fig. (9): Excel Program Sheet for Numerical Example 3

X. CONCLUSION

In conducting a mechanism synthesis or parameter study, a structural error may occur. This refers to the inability of a mechanism to generate precisely certain function values at certain precision points, even though certain function values can be generated over a continuous range [7]. In the numerical example 1 program results, compare Fig. (6) and Fig. (7), and make a note of the crank angles, 0º, 180º, and 360º; whilst at 0º, a division by zero results in the error, #DIV/0!, at 180º and 360º, no such error is recognized and displayed by the program. Also the value for crank pin velocity, computed with (6) does not near tally with the value obtained, using (6a). To avoid such error, the angles 0.1º, 179.99º, 359.99º, have been selected as the starting, middle and end points for the full cycle of the inline model. Though, a full cycle program analysis is shown, for the inline model, this is not necessary, since the model exhibits symmetry – an antisymmetric velocity plot. The model can thus be programmed for the half-cycle-0º-to-180º. Note also that for the inline model the (V_p/V_B) and ($V_p/\omega r$) versus crank angle plots align. The offset model velocity plot in comparison, is not symmetric however, and must be analysed for the full cycle. Inspection of the results of the numerical example 3 in Fig. (9), for the offset model shows that, the crank pin velocity, computed with (6) does not tally with result using (6a) at several points. This may be because the [5] derived offset model assumed a negative connecting rod angle. Equation (6) was derived using a positive connecting rod angle. Correcting for that by inputting a negative connecting rod angle in (6), results in the second numerical example 3 results displayed in Fig. (10), with equal, but negative V_B values. The (V_p/V_B) and ($V_p/\omega r$) versus crank angle response plots are also opposed, as earlier obtained in the Fig. (9) results. It is not clear if this may have been as a result of the choice of direction of rotation selected in the [5] model.

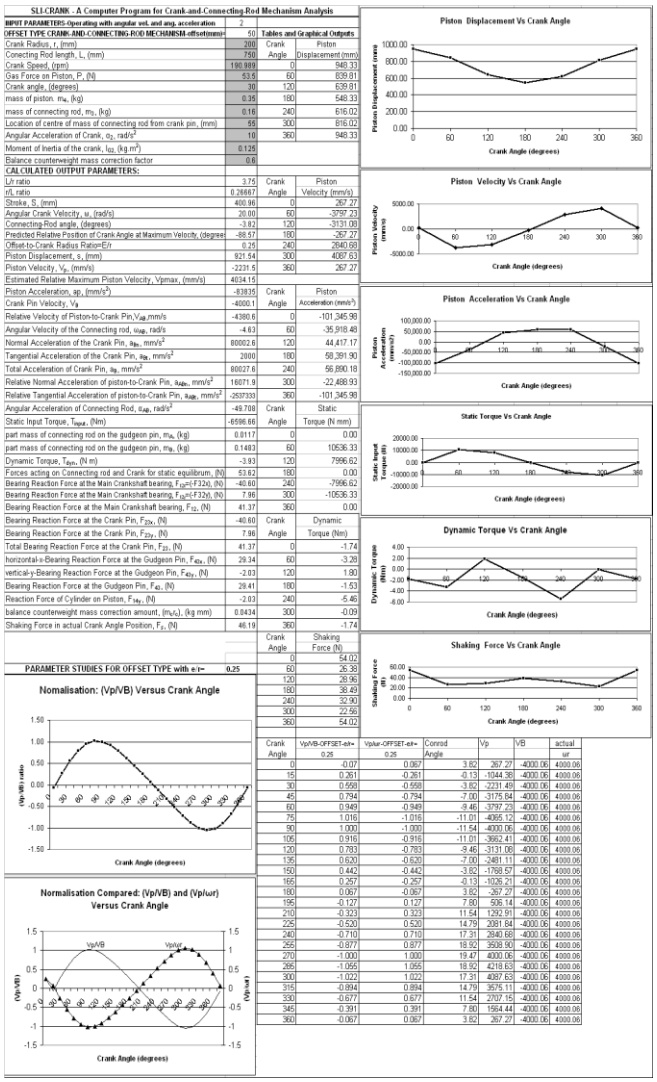

Fig. (10): Numerical Example 3 with adjusted connecting rod angle sign

APPENDIX

From equation (13d),

$$r\alpha Cos\theta - r\omega^2 Sin\theta = L\omega_{AB}^2 \qquad (13d)$$

$$\omega^2{}_{AB} = \frac{r}{L}\alpha Cos\theta - \frac{r}{L}\omega^2 Sin\theta \qquad (A1)$$

$$\omega_{AB} = \left[\frac{r}{L}\alpha Cos\theta - \frac{r}{L}\omega^2 Sin\theta\right]^{\frac{1}{2}} \qquad (A2)$$

For maximum stationary point condition:

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$$\frac{d\omega_{AB}}{d\theta} = \frac{1}{2}\left[\frac{r}{L}\alpha(-Sin\theta) - \frac{r}{L}\omega^2(Cos\theta)\right]^{-\frac{1}{2}} = 0 \qquad \text{(A3)}$$

Simplifying,

$$-\alpha Sin\theta = \omega^2 Cos\theta \qquad \text{(A4)}$$

Or

$$\tan\theta = \left(\frac{-\omega^2}{\alpha}\right) \qquad \text{(A5)}$$

Nomenclature

a_p	Piston acceleration [mm/s^2]
a_{Bn}	Normal crank pin acceleration [mm/s^2]
a_{Bt}	Tangential crank pin acceleration [mm/s^2]
a_{ABn}	Relative normal acceleration of piston to crank pin [mm/s^2]
a_{ABt}	Relative tangential acceleration of piston to crank pin [mm/s^2]
e	Offset or Eccentricity [mm]
f_d	Design factor [mm]
F_{12}	Bearing Reaction Force at the Main Crankshaft bearing [N]
F_{14y}	Reaction Force of Cylinder on Piston [N]
F_{23}	Bearing Reaction Force at the Crank Pin [N]
F_{32}	$=F_{34}$ = Forces acting on Connecting rod and Crank for static equilibrium [N]
F_{43}	Bearing Reaction Force at the Gudgeon Pin [N]
I_{G2}	Moment of Inertia of Crankshaft, [kg.m^2]
I_{G3}	Moment of Inertia of Connecting rod, [kg.m^2]
L	Connecting rod length [mm]
m_4	Piston mass, [kg]
m_3	Connecting rod mass, [kg]
m_A	Equivalent Connecting rod mass on gudgeon pin at piston end, [kg]
m_B	Equivalent Connecting rod mass on Crank pin, [kg]
m_c	Balance mass, [kg]
r_c	Distance of balance mass from main bearing [mm]
r	Crank radius [mm]
s	Piston displacement [mm]
v	Piston velocity = V_A [mm/s]
V_B	Crank pin velocity [mm/s]
V_{AB}	Relative velocity of piston to crank pin [mm/s]
β	Connecting rod angle [degrees]
θ	Crank angle [degrees]
ω	Crank angular velocity [rad/s]
ω_{AB}	Angular velocity of connecting rod [rad/s]
α	Crank angular acceleration [rad/s^2]
α_{AB}	Angular acceleration of connecting rod [rad/s^2]
P	Combustion Gas Force on Piston [N]
S_t	Stroke length (mm)
T_{input}	Static Input Torque of crankshaft [Nmm]
$T_{dynamic}$	Dynamic Torque [Nmm]

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AUTHOR PROFILE

Tonye K. Jack is a Registered Engineer, and ASME member. He worked on plant maintenance and rotating equipment in the Chemical Fertilizer industry, and on gas turbines in the oil and gas industry. He has Bachelors degree in Mechanical Engineering from the University of Nigeria, and Masters Degrees in Engineering Management from the University of Port Harcourt, and in Rotating Machines Design from the Cranfield University in England. He is a University Teacher in Port Harcourt, Rivers State, Nigeria, teaching undergraduate classes in mechanical engineering. His research interests are on rotating equipment engineering, maintenance, engineering management, engineering computer programs, and applied mechanics.

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